JOHN R. APEL

A LINEAR RESPONSE THEORY FOR WAVES IN A GEOPHYSICAL FLUID

Linear waves in the ocean and atmosphere occur in a wide variety of types, with periods ranging from microseconds to months. A unified theoretical treatment of such waves is developed, and a general solution for the fields is presented. Some of the propagation characteristics are summarized in terms of a "parameter fluid" that shows allowed and forbidden regions of wave motion as the frequency, buoyancy, and rotation rate are varied.

INTRODUCTION

Waves in a planetary or geophysical fluid occur in a rich and sometimes confusing variety of types. The classification of a given wave according to generic type (i.e., acoustic, gravity, planetary, etc.) is usually made on the basis of frequency, although in some cases the assignment to one or another of these families is not a clear-cut procedure.

A unified treatment of linearized waves in a rotating, stratified fluid has several advantages. First, from the purely theoretical view, an analysis that yields essentially all waves in a single equation is a more comprehensive and didactic theory, although it is certainly more complicated. Second, if coupling between various wave modes is to be studied (as, for instance, might be the case between infrasonic and high-frequency internal waves or between low-frequency gravity and planetary waves), a scale analysis must be generalized accordingly. Third, all approximation schemes suffer from difficulties in assaying their ranges of applicability, a malaise that is avoided by the more general treatment.

In several ways, the theoretical formulation is similar to ones used in the electrodynamics of continuous media; fluid dynamics and electrodynamics have well-known parallels, of course. The insight provided by the equivalences is useful in understanding the nature of fluid motion.

As a model, we take a stratified, compressible, single component fluid, flowing with uniform horizontal velocity on a rotating beta plane in a gravitational field that is directed normal to the plane. Eddy viscosity and heat conductivity are treated by introducing anisotropic diffusion and heat-flow coefficients. The thermodynamic properties of the fluid are introduced through equations for thermal and internal energy, entropy, and heat flow. Stratification is described by the Brunt-Väisälä frequency, compressibility by the acoustic speed, and rotational force by a Coriolis frequency linearly varying in the north-south direction.

Boundary conditions are purposefully kept general in order to separate those features of the motion that

SYNOPSIS

"Geophysical fluid" is the name applied to the oceanic and atmospheric fluid envelopes that bathe the earth. One liquid and one gaseous, their dynamics may be distinguished from that of ordinary fluids by several characteristics, chiefly strong vertical stratification, rapid rotation due to the earth's spin, and small thickness relative to their horizontal size. From the theoretical standpoint, both can be described by the same set of mechanical and thermodynamical equations, with essentially only their equations of state and their upper boundary conditions being different.

There are several distinct types of waves in geophysical fluids, differentiated by the restoring forces acting on a displaced parcel of fluid. Compressibility gives rise to acoustic body waves, surface tension to capillary waves, gravity to surface gravity waves, gravity plus buoyancy to internal waves, Coriolis force to inertial oscillations, and variation in Coriolis force to Rossby or vorticity waves. In addition, the presence of boundaries introduces other edge modes, such as Kelvin and Lamb waves.

While descriptions of all of these oscillations are contained in the equations cited, they are usually separated out and treated one by one. For the case of fluid equations having constant coefficients, the present treatment attempts to derive a single solution in terms of Fourier-Laplace inversion integrals that describe all classes of linear waves, as well as a single dispersion equation relating frequency to wave vector. The necessarily complicated algebra is somewhat simplified by the introduction of several characteristic frequencies, spatial scales, and auxiliary quantities such as the vector index of refraction that aid in the description. Another aid is termed the "parameter fluid," which is a graphical technique for distinguishing between the propagation features of the various modes. By these means, many of the known types of linear waves are shown to be contained in the fomulation presented. Future work will attempt to generalize the formulation and to present a solution to the dispersion relation that describes all of the wave modes supported by the fluid, as well as their possible interactions.

are caused by the intrinsic properties of the fluid from those caused by boundary effects. The present formulation is given in terms of rectangular boundaries, and the solution for the velocity field describes waves ranging through planetary, inertial, internal, gravity, and acoustic frequencies, including, of course, coupling between these different types of oscillations.

Several important results derive from the calculations. First, a solution of considerable generality is obtained for the first-order velocity field, formulated in terms of integrals involving mixed initial and boundary conditions and a linear response function for the fluid. Second, a generalized dispersion equation is derived governing the relationship between frequency and wave number components for oscillations ranging from acoustic to Rossby wave frequencies. Several familiar cases of dispersion equations are recovered from the general case. Third, the concepts of a complex vector index of refraction and a vector impedance, which are applicable throughout the total frequency range of the waves, are introduced; these entities suggest methods of using, in geophysical wave dynamics, certain mathematical techniques such as extremum principles or theories of wave propagation in random media. Fourth, the idea of a fluid parameter space is advanced; it shows the behavior of the index of refraction as a function of parameters involving the characteristic frequencies and scales of the system. A graphical example of a parameter space is given for an exponential atmosphere.

THE BASIC APPROACH

The basic approach is to decompose an arbitrary initial disturbance in the fluid into its temporal and spatial frequency components; next, propagate those components through the medium; and then reassemble them at the point of interest in a way that shows the modifications to the waves introduced by the fluid and the boundaries. From the Green's function, a generalized dispersion *relation* is derived whose roots then give the dispersion *equations* for the various branches or modes—acoustic, gravity, etc. It is shown that (a) such dispersion equations generally hold only in the long-time limit when the arbitrary initial excitation has died away and only the free waves remain; and (b) for the general initial-value problem, the initial conditions contribute strongly in the short-time limit.

The vantage point of a linear response formulation allows one to view the situation as an input-output problem, with the excitation that occurs at one time and place in the fluid propagating through the medium to other places at later times in a way described by the tensor Green's function, $\mathbf{D}^{-1}(\mathbf{k},\omega)$.* This is shown schematically in Fig. 1.

A more complete development of the theory presented here may be found in Ref. 1, along with an extension that includes an explicit treatment of Rossby waves. Because of space limitations, that discussion has not been included here.

*A glossary of symbols appears at the end of this article.

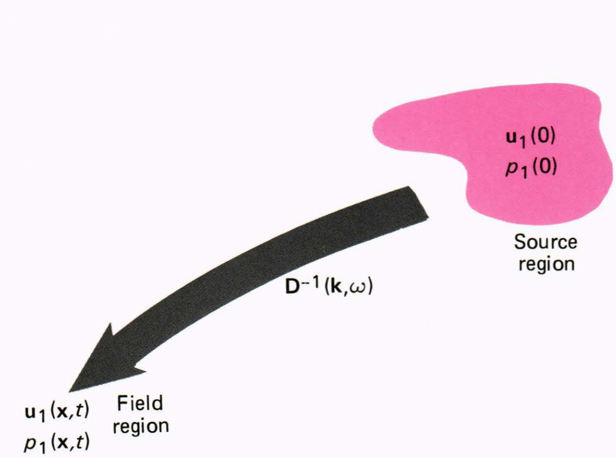

Figure 1—The initial values of velocity, $\mathbf{u}_1(0)$, and pressure, $p_1(0)$, are specified in some source region of the fluid; these quantities then propagate away and are modified by the properties of the fluid as given by the linear transfer function, $\mathbf{D}^{-1}(\mathbf{k},\omega)$, to appear at position $\mathbf{x}$ at time t as field variables $\mathbf{u}_1(\mathbf{x},t)$ and $p_1(\mathbf{x},t)$.

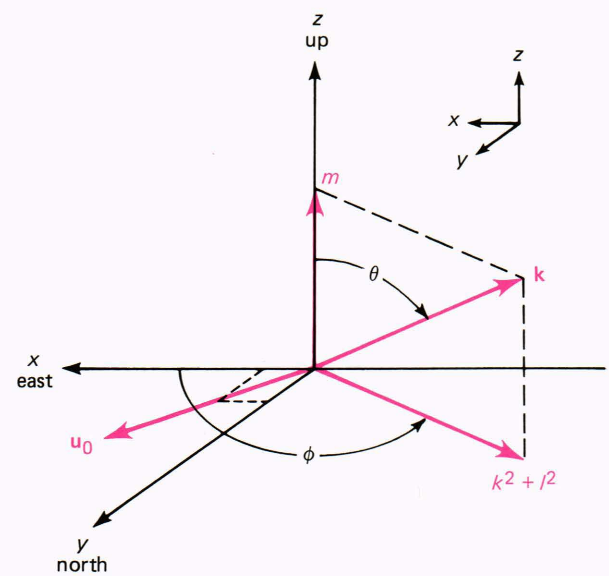

Figure 2—The notation that is used in the tangent-plane coordinate system.

SYSTEMS OF EQUATIONS

Basic Equations

The basic equations of dynamics and thermodynamics used below are appropriate to a single component fluid; they constitute nine equations in the nine dependent variables: velocity ($\mathbf{u} = u\hat{x} + v\hat{y} + w\hat{z}$), specific volume ($\alpha = 1/\rho$), temperature ($T$), pressure ($p$), entropy ($s$), heat ($q$), and internal energy ($e$) per unit mass. Equations 1 through 9 define the notation used. In each equation, $d/dt = \partial/\partial t + \mathbf{u}\cdot\nabla$ represents the convective derivative. Figure 2 shows the tangent-plane coordinate system used in the development of the theory.

Momentum:

$$\frac{\partial u}{\partial t} + \mathbf{u}\cdot\nabla u - \Omega v \sin\lambda + \Omega w \cos\lambda = -\alpha \frac{\partial p}{\partial x} + \nabla\cdot\mathsf{A}\cdot\nabla u\,, \quad (1)$$

$$\frac{\partial v}{\partial t} + \mathbf{u}\cdot\nabla v + \Omega u \sin\lambda = -\alpha \frac{\partial p}{\partial y} + \nabla\cdot\mathsf{A}\cdot\nabla v\,, \quad (2)$$

$$\frac{\partial w}{\partial t} + \mathbf{u}\cdot\nabla w - \Omega u \cos\lambda = -\alpha \frac{\partial p}{\partial z} + \nabla\cdot\mathsf{A}\cdot\nabla w - g\,. \quad (3)$$

Continuity:

$$\frac{d\alpha}{dt} - \alpha\nabla\cdot\mathbf{u} = 0\,. \quad (4)$$

First Law of Thermodynamics:

$$\frac{de}{dt} = \frac{dq}{dt} - p\frac{d\alpha}{dt}\,. \quad (5)$$

Second Law of Thermodynamics:

$$\frac{ds}{dt} = \frac{1}{T}\frac{dq}{dt}\,. \quad (6)$$

Heat flow:

$$\rho\frac{d}{dt}(C_\alpha T) = \rho\frac{dq}{dt} = \nabla\cdot\mathsf{K}\cdot\nabla T\,. \quad (7)$$

Equation of state:

$$\rho = \rho(p,T)\,. \quad (8)$$

Internal energy:

$$e = e(\alpha,s)\,. \quad (9)$$

The remaining quantities in Eqs. 1 through 9 are the tensor eddy viscosity, A, which has the matrix representation

$$\mathsf{A} = \begin{bmatrix} A_h & 0 & 0 \\ 0 & A_h & 0 \\ 0 & 0 & A_v \end{bmatrix}, \quad (10)$$

and the eddy heat conductivity, K, which has a similar form; both reflect the anisotropy between the horizontal (A_h, K_h) and vertical (A_v, K_v) components of the momentum and heat diffusion tensors in the ocean and atmosphere. These terms represent a phenomenological description of complicated turbulent diffusion processes and are not rigorous. The numerical values of A and K depend on the length scales in which one is interested; similarly, the scales of motion selected determine whether the diffusivities are themselves functions of position.

The introduction of several thermodynamic relationships and thermomechanical coefficients allows one to avoid the use of an explicit equation of state and to eliminate further reference to the internal energy. Continuing so as to define the notation, these are written as

Pressure:

$$p = -(\partial e/\partial\alpha)_s\,, \quad (11)$$

Temperature:

$$T = (\partial e/\partial s)_\alpha\,, \quad (12)$$

Sound speed:

$$c = \sqrt{(\partial p/\partial\rho)_s}\,, \quad (13)$$

Coefficient of thermal expansion:

$$a = (\partial\alpha/\partial T)_p/\alpha\,, \quad (14)$$

Specific heat at constant volume:

$$C_\alpha = (\partial e/\partial T)_\alpha\,, \quad (15)$$

Specific heat at constant pressure:

$$C_p = T(\partial s/\partial T)_p\,, \quad (16)$$

Ratio of specific heats:

$$\gamma = C_p/C_\alpha\,. \quad (17)$$

Following Eckart,[2] the thermodynamic identity is used:

$$C_\alpha\gamma(\gamma - 1) = a^2c^2T\,. \quad (18)$$

Zero-Order Equations

The zero-order variables are assumed constant in time, with all of the coefficients of the zero-order equations except $\mathbf{\Omega}$ being constant in space. The first-order motions represent departures from this state.

The dependent variables are expanded in a perturbation series and, by the usual means, one obtains the zero-order equations, written for constant, horizontal mean flow, $\mathbf{u}_0 = (u_0, v_0, 0)$:

$$\mathbf{\Omega}\times\mathbf{u}_0 = -\alpha_0\nabla p_0 - g\hat{z}\,, \quad (19)$$

$$\mathbf{u}_0\cdot\nabla\alpha_0 = 0\,, \quad (20)$$

$$\mathbf{u}_0 \cdot \nabla s_0 = \frac{1}{T_0} \dot{q}_0 = 0 , \tag{21}$$

$$\alpha_0 \nabla \cdot \mathsf{K} \cdot \nabla T_0 = \dot{q}_0 = 0 . \tag{22}$$

Equation 19 contains the equation for geostrophic flow as its horizontal component and the hydrostatic equation as its vertical component. Equation 20 requires the density gradient to be normal to the horizontal mean flow, i.e., vertical.

If ρ_0 is assumed to vary in the vertical only, one obtains for the pressure gradient

$$\nabla p_0 = -\rho_0(\mathbf{\Omega} \times \mathbf{u}_0 + g\hat{z}) . \tag{23}$$

Analogous expressions for entropy and temperature gradients may be obtained but are not given here for brevity.

We now introduce two buoyancy frequencies for later convenience. The first is termed the Brunt-Väisälä frequency, N_z:

$$N_z^{\,2} \equiv -g\left(\frac{1}{\rho_0}\frac{d\rho_0}{dz} + \frac{g}{c^2}\right) , \tag{24}$$

whose role in establishing buoyancy oscillations is well understood. A second buoyancy frequency, N_a, is defined via

$$N_a = -\frac{c}{2\rho_0}\frac{d\rho_0}{dz} . \tag{25}$$

In the meteorological literature, this quantity is often called the atmospheric cutoff frequency, since it is the lowest frequency at which acoustic waves can propagate in an exponential atmosphere.

Assume now that the Coriolis vector, $\mathbf{\Omega}$, has only a vertical component and can be expanded in a series in the north-south direction:

$$|\mathbf{\Omega}| = 2\,\Omega_E \sin\lambda \equiv f \approx f_0 + \beta y , \tag{26}$$

where f is the Coriolis parameter, β is the meridional derivative, Ω_E is the angular speed of the planet, and λ is the latitude. Next, define a reduced gravitational acceleration, $g\boldsymbol{\xi}$, whose horizontal components are proportional to the slopes of isopycnal surfaces in the fluid and whose vertical component is $-g$. Let $\eta(x,y)$ define such a surface. Then, from the geostrophic condition, one obtains

$$\xi_x \equiv \frac{fv_0}{g} = \frac{\partial\eta}{\partial x} \tag{27}$$

$$\xi_y \equiv -\frac{fu_0}{g} = \frac{\partial\eta}{\partial y} . \tag{28}$$

The vector $\boldsymbol{\xi}$ is then defined as

$$\begin{aligned}\boldsymbol{\xi} &\equiv -\frac{1}{g}(\mathbf{\Omega} \times \mathbf{u}_0 + g\hat{z}) \\ &\equiv \xi_x\,\hat{x} + \xi_y\,\hat{y} - \hat{z} .\end{aligned} \tag{29}$$

With these abbreviations, the zero-order pressure gradient may be written as

$$\nabla p_0 = \rho_0 g\boldsymbol{\xi} . \tag{30}$$

First-Order Equations

In the eight first-order equations, the total time derivative of a linearized variable, $\psi = \psi_0 + \psi_1$, has the form

$$d\psi/dt = \partial\psi_1/\partial t + \mathbf{u}_0 \cdot \nabla\psi_1 + \mathbf{u}_1 \cdot \nabla\psi_0 , \tag{31}$$

where products of first-order terms have been neglected. The momentum equation,

$$\begin{aligned}\partial\mathbf{u}_1/\partial t + \mathbf{u}_0 \cdot \nabla\mathbf{u}_1 + \mathbf{\Omega} \times \mathbf{u}_1 - \nabla \cdot \mathsf{A} \cdot \nabla\mathbf{u}_1 \\ + \alpha_0 \nabla p_1 + \alpha_1 \nabla p_0 = \mathbf{0} ,\end{aligned} \tag{32}$$

contains advective, Coriolis, eddy viscosity, pressure, and buoyancy terms, with gravity implicitly appearing in the term ∇p_0. Velocity-shear terms $(\nabla \cdot \mathbf{u}_0)$ do not appear because of the assumption of a uniform velocity field. Conservation of mass is assured to first order by writing the continuity equation for the specific volume:

$$\partial\alpha_1/\partial t + \mathbf{u}_0 \cdot \nabla\alpha_1 + \mathbf{u}_1 \cdot \nabla\alpha_0 - \alpha_0 \nabla \cdot \mathbf{u}_1 = 0. \tag{33}$$

The Second Law of Thermodynamics becomes

$$\partial s_1/\partial t + \mathbf{u}_0 \cdot \nabla s_1 + \mathbf{u}_1 \cdot \nabla s_0 = \dot{q}_1/T_0 , \tag{34}$$

the heat flow equation is

$$\nabla \cdot \mathsf{K} \cdot \nabla T_1 = \rho_0 \dot{q}_1 , \tag{35}$$

and the first-order thermodynamic equations replacing the first law (equation of state and internal energy) are

$$p_1 = -\rho_0^2 c^2 \alpha_1 + \frac{\rho_0(\gamma - 1)}{a} s_1 , \tag{36}$$

$$T_1 = -\frac{\rho_0(\gamma - 1)}{a}\alpha_1 + \frac{T_0}{C_\alpha} s_1 . \tag{37}$$

(See Apel[1] and Eckart[2] for the derivation of Eqs. 36 and 37.)

By applying Eq. 31 to the time-differentiated versions of Eqs. 36 and 37 and then using Eq. 34 to eliminate ds_1/dt, one obtains for the pressure and temperature, respectively,

$$\partial p_1/\partial t + \mathbf{u}_0 \cdot \nabla p_1 + \mathbf{u}_1 \cdot \nabla p_0 + \rho_0 c^2 \nabla \cdot \mathbf{u}_1 = (\rho_0 a c^2 / C_p)\dot{q}_1 \quad (38)$$

and

$$\partial T_1/\partial t + \mathbf{u}_0 \cdot \nabla T_1 + \mathbf{u}_1 \cdot \nabla T_0 + [(\gamma - 1)/a] \nabla \cdot \mathbf{u}_1 = \dot{q}_1 / C_\alpha . \quad (39)$$

The subsequent development of the theory will take place using Eqs. 32 to 35, 38, and 39 for the first-order quantities and Eq. 30 for the zero-order gradient.

Eckart Field Equations

Eckart recognized the value of transforming the first-order quantities by using the acoustic impedance, Z_0, to scale out the (approximately) exponential density variation with height. That impedance is

$$Z_0(z) = \rho_0(z) c . \quad (40)$$

We will follow this procedure for the first-order fields, using slight variations from the Eckart definitions; a capital letter will denote the transformed version of a lower case quantity, wherever possible.

Velocity field:

$$\mathbf{U}(\mathbf{x},t) = (U,V,W) = \mathbf{u}_1(\mathbf{x},t)\sqrt{\rho_0 c} . \quad (41)$$

Density and specific volume fields:

$$R(\mathbf{x},t) = \rho_1(\mathbf{x},t) / \sqrt{\rho_0 c} , \quad (42a)$$

$$A(\mathbf{x},t) = \alpha_1(\mathbf{x},t) \sqrt{\rho_0 c} . \quad (42b)$$

Pressure field:

$$P(\mathbf{x},t) = p_1(\mathbf{x},t) / \sqrt{\rho_0 c} . \quad (43)$$

Entropy field:

$$S(\mathbf{x},t) = s_1(\mathbf{x},t)\, g(\gamma - 1) \sqrt{\rho_0 c} / a c^2 . \quad (44)$$

Temperature field:

$$T'(\mathbf{x},t) = T_1(\mathbf{x},t)\, a \sqrt{\rho_0 c} / \gamma . \quad (45)$$

Heating rate field:

$$Q(\mathbf{x},t) = \dot{q}_1(\mathbf{x},t)\, g(\gamma - 1) \sqrt{\rho_0 c} / a c^2 T_0 . \quad (46)$$

The relations obeyed by the Eckart fields are obtained by substituting their defining relations into the first-order equations (in the derivatives operating on A and K, terms of order Γ_0^2 have been neglected):

$$\partial \mathbf{U}/\partial t + \mathbf{u}_0 \cdot \nabla \mathbf{U} + (\nabla - \Gamma_0 \hat{z}) \cdot \mathsf{A} \cdot (\nabla - \Gamma_0 \hat{z}) \mathbf{U} + \mathbf{\Omega} \times \mathbf{U} = -c(\nabla + \Gamma) P - S\xi , \quad (47)$$

$$\partial A/\partial t + \mathbf{u}_0 \cdot \nabla A = \alpha_0 (\nabla + \Gamma_0 \hat{z}) \cdot \mathbf{U} , \quad (48)$$

$$\partial S/\partial t + \mathbf{u}_0 \cdot \nabla S = Q - \mathbf{N}^2 \cdot \mathbf{U} , \quad (49)$$

$$\partial P/\partial t + \mathbf{u}_0 \cdot \nabla P = cQ/g - c(\nabla - \mathbf{\Gamma}) \cdot \mathbf{U} , \quad (50)$$

$$\partial T'/\partial t + \mathbf{u}_0 \cdot \nabla T' = Q/g + [\gamma^{-1}(\nabla + \Gamma_0 \hat{z}) - (\nabla - \mathbf{\Gamma})] \cdot \mathbf{U} , \quad (51)$$

$$(\nabla - \Gamma_0 \hat{z}) \cdot \mathsf{K} \cdot (\nabla - \Gamma_0 \hat{z}) T' = \rho_0 C_\alpha Q/g . \quad (52)$$

In the course of taking spatial derivatives, the vertical variation in $\rho_0(z)$ generates three terms that are essentially reciprocal scale heights, which are introduced for convenience of notation. The first, Γ_0, is

$$\Gamma_0 \equiv \frac{1}{2\rho_0} \frac{d\rho_0}{dz} \quad (53)$$

and will be termed the transition attenuation coefficient. The second, Γ_g, is the reciprocal scale height for compressibility:

$$\Gamma_g \equiv -g/c^2 . \quad (54)$$

The third, Γ_z, is defined by

$$\Gamma_z \equiv \Gamma_0 - \Gamma_g = \frac{1}{2\rho_0} \frac{d\rho_0}{dz} + \frac{g}{c^2} \quad (55)$$

and is Eckart's vertical attenuation or adiabatic coefficient. A useful identity that relates the Brunt-Väisälä and atmospheric cutoff frequencies to Γ_z and Γ_0 is

$$N_a^2 = N_z^2 + \Gamma_z^2 c^2 = (-\Gamma_0 c)^2 . \quad (56)$$

From Eqs. 53 and 55 evaluated for the case of constant coefficients, the density is

$$\rho_0(z) = \rho_0(z_0) \exp[2(\Gamma_z + \Gamma_g)(z - z_0)] . \quad (57)$$

When the stratification $\rho_0'/2\rho_0$ dominates over Γ_g, as is usually the case in slightly compressible fluids, Γ_0^{-1} is essentially the characteristic scale of the gradient. Its order of magnitude is 150 kilometers in the upper ocean, whereas Γ_g^{-1} is approximately 225 kilometers.

The Coriolis force also introduces reciprocal lengths for horizontal motions, which we define as the baroclinic attenuation coefficients, Γ_x and Γ_y:

$$\Gamma_x \equiv -fv_0/c^2 \equiv -N_x^2/g \, , \tag{58a}$$

$$\Gamma_y \equiv fu_0/c^2 \equiv -N_y^2/g \, . \tag{58b}$$

These quantities are analogous to reciprocal Rossby radii of deformation, f/c, for a fluid in which the limiting velocity is c, scaled by the Mach number, u_0/c. The order of magnitude for Γ_x^{-1} in the ocean, as defined, is 10^{10} meters.

In the present limit of an infinitely deep fluid, it is the acoustic speed, c, that establishes the limiting velocity for waves, and in this regard, c plays the role of the velocity of light in electromagnetic theory. Later on, it will be shown that in a single-layer system of depth H, the waveguide effect presented by a rigid bottom and deformable top surface constrains the allowed values of m to be very small wave numbers. In this shallow-water, slow-wave system, the square of the acoustic speed, c^2, is harmonically summed with gH to give an equivalent speed, c_e, via

$$\frac{1}{c^2} + \frac{1}{gH} \equiv \frac{1}{c_e^2} \, , \tag{59}$$

which, when $c^2 \gg gH$, allows the neglect of acoustic effects on the wave speed in all but the deepest ocean, excepting, of course, for sound waves. Thus, neglect of Γ_x and Γ_y is not justified in a shallow fluid, and indeed, those quantities become Rossby radii of deformation in a bounded fluid that is in geostrophic adjustment.

These reciprocal lengths are summarized by the vector $\mathbf{\Gamma}$, termed simply the attenuation vector:

$$\mathbf{\Gamma} \equiv \Gamma_x \hat{x} + \Gamma_y \hat{y} + \Gamma_z \hat{z} = \frac{f}{c^2}\,(-v_0 \hat{x} + u_0 \hat{y}) + \left(\frac{1}{2\rho_0}\frac{d\rho_0}{dz} + \frac{g}{c^2}\right)\hat{z} \, . \tag{60}$$

The role of $\mathbf{\Gamma}$ and $\Gamma_0 \hat{z}$ in the field equations is to introduce attenuation or amplification terms in the spatial derivatives, which arise as a result of Coriolis forces and stratification. The neglect of Γ_0 and g/c^2 is equivalent to the Boussinesq approximation.

Finally, the Γ's combine to form a vector buoyancy parameter, $\mathbf{N}^2$, defined as

$$\mathbf{N}^2 \equiv N_x^2 \hat{x} + N_y^2 \hat{y} + N_z^2 \hat{z} \equiv \left(\frac{fg}{c^2}\right)(v_0 \hat{x} - u_0 \hat{y}) - g\left(\frac{1}{\rho_0}\frac{d\rho_0}{dz} + \frac{g}{c^2}\right)\hat{z} \, . \tag{61}$$

THE FORMAL SOLUTION

In order to arrive at the complete solution for the Eckart fields, replete with initial and boundary values, we will decompose those quantities into their frequency and wave number components by performing a Laplace transform, $\mathfrak{L}$, followed by a finite Fourier transform, $\mathfrak{F} = \mathfrak{F}_x \mathfrak{F}_y \mathfrak{F}_z$, in all three space variables. The same symbol will be used for both the space-time field and its wave number-frequency transform, with the functional arguments indicating just which of the four independent variables has been transformed. Thus we take

$$\mathbf{U}(\mathbf{k},\omega) = \mathfrak{F}\mathfrak{L}[\mathbf{U}(\mathbf{x},t)] = \int_{\mathbf{x}_1}^{\mathbf{x}_2} d\mathbf{x} \int_0^\infty dt \, \exp[-i(\mathbf{k}\cdot\mathbf{x} - \omega t)]\mathbf{U}(\mathbf{x},t). \tag{62}$$

The space-time behavior of the field variables may be taken as

$$\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)] \, , \tag{63}$$

where the total phase is

$$\mathbf{k}\cdot\mathbf{x} - \omega t = kx + ly + mz - (\omega_r + i\omega_i)t \, , \tag{64}$$

thereby defining the (x,y,z) components of wave vector, $\mathbf{k}$, as

$$\mathbf{k} = (k,l,m)$$

and the complex frequency, ω, as

$$\omega = \omega_r + i\omega_i \, . \tag{65}$$

The Laplace transform variable has been taken as $i\omega = i\omega_r - \omega_i$ rather than $-s$, the usual symbol, in order to retain the conventional notation for plane waves used above. The Laplace inversion integral is thereby evaluated along a modified path in a way to be discussed below. The spatial integral is over all three coordinates.

Integral transforms are most useful for equations having constant coefficients; upon transformation, a linear coefficient in a spatial variable generates a derivative in the conjugate wave number variable. In the present case, $\mathbf{\Omega}$, via f, varies linearly in y; in addition, the parameters $\mathbf{N}^2$, ξ, and $\mathbf{\Gamma}$ contain f. Thus the effect of the y-Fourier transform is to generate second-order differential equations in y-wave number space that are scarcely simpler than the originals. The exception appears to be when $\mathbf{u}_0 = \mathbf{o}$, in which case only the $\mathbf{\Omega} \times \mathbf{U}$ term persists; it produces differential equations solvable in terms of known functions.

For this reason, the development of the theory here reaches a branch point. If the mean flow is to be included, the beta effect must be taken as zero, and vice versa. The case of the variable Coriolis parameter is complicated and will not be treated here (see Ref. 1 for a more complete discussion). Instead, only the constant-f case will be developed.

In applying the fourfold transformations $\mathfrak{F}$ and $\mathfrak{L}$, their effects on the convective derivative are to generate the Doppler-shifted frequency,

$$\omega_d \equiv \omega - \mathbf{u}_0 \cdot \mathbf{k} \ , \quad (66)$$

plus mixed initial and boundary values of the dependent variables.

The presence of $\alpha_0(z)$ and $\rho_0(z)$ in Eqs. 48 and 52 has been dealt with by using the exponential density, Eq. 57, with the constant Γ_0 as the scale height. Upon Fourier transformation in z, this factor results in making the z-component of wave vector complex.

We now define a number of parameters that will allow the solution to be written somewhat more compactly.

Indices of Refraction. We define the complex vector indices of refraction, $\mathbf{n}$ and $\mathbf{n}_0$, via

$$\mathbf{n} \equiv (\mathbf{k} - i\boldsymbol{\Gamma}) \ c/\omega_d \ . \quad (67a)$$

The adjoint index (considered to be a row matrix) is

$$\mathbf{n}^+ \equiv (\mathbf{k} + i\boldsymbol{\Gamma})c/\omega_d \ . \quad (67b)$$

Also, an analogous quantity appearing in the eddy viscosity terms is $\mathbf{n}_0$, where

$$\mathbf{n}_0 \equiv (\mathbf{k} - i\Gamma_0\hat{z})c/\omega_d \ . \quad (68)$$

The justification for designating these as indices of refraction is as follows. The normal scalar index of refraction is the ratio of some reference speed (in this case, the acoustic speed, c) to the phase speed of the wave. The acoustic speed provides an absolute scale for velocity in the theory, much as does the velocity of light in electromagnetic theory. The fluid wave phase speed is

$$c_\phi = \frac{\omega(\mathbf{k})}{|\mathbf{k}|} \ , \quad (69)$$

and it is thus proper to assign the direction of the wave vector to the index of refraction. As a further generalization, a wave propagating in a current moving at $\mathbf{u}_0$ is altered in speed and direction by the Doppler shift of the current, which thus acts to refract the wave. Hence a reasonable definition of a vector index of refraction in a moving medium might be

$$\mathbf{n}(\boldsymbol{\Gamma} = \mathbf{o}) = \frac{\mathbf{k} \ c}{\omega - \mathbf{u}_0 \cdot \mathbf{k}} \ . \quad (70)$$

However, the terms involving $\boldsymbol{\Gamma}$ and $\Gamma_0\hat{z}$ arise in the theory in essentially the same way as do those involving $\mathbf{k}$, except for their multiplication by i. These terms, appearing as real phase factors in the exponent, therefore describe attenuation or amplification due to the horizontal and vertical variability in the zero-order properties of the medium. Thus the definitions (Eqs. 67 and 68) may be considered as generalized vector indices of refraction describing oscillatory, evanescent, and amplifying waves.

The vector index of refraction is proportional to a quantity called the *slowness* of the wave, $\mathbf{s} = \mathbf{k}/\omega$, a vector whose components are the reciprocals of the phase speeds in the (x,y,z) directions. Its projections on the coordinate axes are

$$\mathbf{s}\cdot\hat{x}_i = k_i/\omega \equiv 1/c_{\phi i}, \quad i = x,y,z \ , \quad (71)$$

whereas a definition of phase *velocity* (such as $\mathbf{c}_\phi = \omega\mathbf{k}/|\mathbf{k}^2|$) has projections that are not the components $c_{\phi i}$ above. The slowness (and the index of refraction) are therefore to be preferred as vector descriptors of wave phase velocity.

Generalized Impedance. An additional aid in interpreting $\mathbf{n}$ is as follows. Neglecting the initial and boundary values, the relationship between the first-order pressure and velocity fields becomes, upon substitution of the defining equations,

$$p_1 = \rho_0 c\mathbf{n}^+ \cdot \mathbf{u}_1$$
$$= \mathbf{Z}\cdot\mathbf{u}_1 \ , \quad (72)$$

where the generalized vector impedance, $\mathbf{Z}$, is

$$\mathbf{Z} \equiv \rho_0 c\mathbf{n}^+ \ , \quad (73)$$

and the dot product is understood as a matrix contraction. The quantity $\rho_0 c$ is the ordinary acoustic impedance; it is natural to define $\mathbf{Z}$ as a generalized impedance for the broader classes of waves discussed here. Thus the refractive index provides a scaling factor not only for the phase velocity of a variety of fluid waves but for their pressure-velocity relationship as well.

The power flux transmitted by the wave is ½ $p_1\mathbf{u}_1^*$ = ½ $\mathbf{Z}\cdot\mathbf{u}_1\mathbf{u}_1^*$, which, in terms of the transformed Eckart fields, becomes ½ $\mathbf{n}^+\cdot\mathbf{UU}^*$. Here again, the index of refraction and impedance display their usefulness.

Wave Parameters. An important geophysical fluid wave parameter is the baroclinic/buoyancy parameter, $\mathbf{B}^2$, defined as

$$\mathbf{B}^2 \equiv \frac{\mathbf{N}^2}{\omega_d^2} = \frac{1}{\omega_d^2}\,[N_x^2 \ N_y^2 \ N_z^2] \equiv [B_x^2 \ B_y^2 \ B_z^2] \ . \quad (74)$$

Its components are the squared baroclinic oscillation frequencies in the horizontal (Eqs. 61) and the squared Brunt-Väisälä frequency in the vertical (Eq. 24) scaled by ω_d^2; $\mathbf{B}$ is a convenient mnemonic symbol for these frequencies. Another parameter is the (constant) Coriolis force parameter, $\mathbf{F}_0$, where

$$\mathbf{F}_0 \equiv \boldsymbol{\Omega}/\omega_d \quad (75)$$

and

$$F_0^2 = f_0^2/\omega_d^2 \ . \quad (76)$$

The eddy and heat diffusion parameters are complex scalars given by

$$\tau \equiv \mathbf{n}_0^+ \cdot \frac{\omega_d \mathsf{A}}{c^2} \cdot \mathbf{n}_0^* \quad (77)$$

and

$$\kappa \equiv \mathbf{n}_0^+ \cdot \frac{\omega_d \mathsf{K}}{c^2} \cdot \mathbf{n}_0^* \ . \tag{78}$$

A considerable simplification of the theory occurs if it is possible to treat the heating rate, $\dot{q}_1$, as a known function, rather than solving for it self-consistently. This has the effect of decoupling the thermodynamic and the hydrodynamic equations and places $\dot{q}$ on the right-hand side of the velocity field equation below.

By algebraic manipulation with those definitions and the elimination of all variables except **U**, one arrives at an important result for the transformed Eckart velocity field:

$$\mathsf{D}(\mathbf{k},\omega)\ \mathbf{U}(\mathbf{k},\omega) = [\mathsf{I} + i\mathbf{F}_0 \times + \ \xi \mathbf{B}^2 \cdot$$

$$- (\mathbf{nn}^+ \cdot \ -i\tau)]\mathbf{U}(\mathbf{k},\omega) = \mathbf{S}(\mathbf{k},\omega) \ . \tag{79}$$

Some discussion of this equation is in order. The vector source function $\mathbf{S}(\mathbf{k},\omega)$ represents initial values specified over all space and boundary values specified over all time, plus the estimate of Q needed to effect the decoupling of the thermodynamic variables just mentioned. Its form can be found in Ref. 1.

Equation 79 defines the tensor dispersion function, D, whose matrix elements are

$$D_{ij} = (1 + i\tau)\delta_{ij} - iF_0 \epsilon_{3ij} + \xi_i B_j^2 - n_i n_j^* \ , \tag{80}$$

where δ_{ij} is the Kronecker index and ϵ_{ijk} and ϵ_{3jk} are permutation indices.

The velocity response of the fluid to the initial and boundary forcing is then summarized by the operator equation

$$\mathsf{D}(\mathbf{k},\omega)\ \mathbf{U}(\mathbf{k},\omega) = \mathbf{S}(\mathbf{k},\omega) \ . \tag{81}$$

The solution for the Eckart field may be obtained formally by defining an inverse operator $\mathsf{D}^{-1}(\mathbf{k},\omega)$ and left-multiplying Eq. 81 by it. This operator is simply the matrix inverse of Eq. 80.

$$\mathbf{U}(\mathbf{k},\omega) = \mathsf{D}^{-1}(\mathbf{k},\omega)\ \mathbf{S}(\mathbf{k},\omega) \ . \tag{82}$$

The solution for the velocity, $\mathbf{u}_1(\mathbf{x},t)$, is obtained immediately from Eq. 82 by Fourier-Laplace inversions and by using Eq. 41 to return to the physical quantities of interest:

$$\mathbf{u}_1(\mathbf{x},t) = (2\pi)^{-4}(\rho_0 c)^{-1/2}$$

$$\times \sum_\lambda \Delta m_\lambda \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \int_{-\infty+i\delta}^{\infty+i\delta} d\omega$$

$$\times \exp\{i[\mathbf{k}\cdot\mathbf{x} - \omega(\mathbf{k})t]\}\ \mathsf{D}^{-1}(\mathbf{k},\omega)\ \mathbf{S}(\mathbf{k},\omega) \ , \tag{83}$$

where the inverse matrix is the sought-for linear response function,

$$\mathsf{D}^{-1}(\mathbf{k},\omega) = \frac{\mathrm{cof}(\mathsf{D})}{\det(\mathsf{D})} \ , \tag{84}$$

and where cof(D) is the cofactor matrix of D and det(D) is its determinant. This, then, is the complete solution to the problem of linear waves in the body and on the surfaces of the fluid, given in terms of the initial, boundary, and heat-flow values, and of a linear response function characterizing the bulk of fluid, D^{-1}, that is independent of initial and boundary values. Equation 83 is a central result of this paper. From its form, the solution may be recognized as essentially the Fourier-Laplace transform of the Green's function for the problem.

A discussion of the inversion integrals in Eq. 83 is in order. The **k**-integration is carried out along the real (k,l,m) axes over the range of accessible wave numbers; if any of the m-values is continuous, as would be caused by boundary conditions that impose such spectra, the associated summation is either supplemented by or replaced with an integral. The ω-integration is along a path in the complex ω-plane, as shown by Contour 1 in Fig. 3, at a distance $\omega_i = \delta$ from, and parallel to, the real ω-axis and above all of the singularities, both poles and branch lines, of $\mathsf{D}^{-1}\mathbf{S}\exp(-i\omega t)$. For $t < 0$, the path is closed in the upper half-plane, where the integrand is analytic and the ω-integral has the value zero, thereby reflecting the principle of causality: no response anywhere prior to $t = 0$. For $t > 0$, it is closed in the lower half-plane, thus encompassing clockwise the singularities due to both the source of excitation and the fluid eigenfrequencies. If complex frequencies or wave numbers are indicated in an instability problem, certain precautions must be observed in performing the integrations. These have been discussed by Briggs in reasonably general terms.[3]

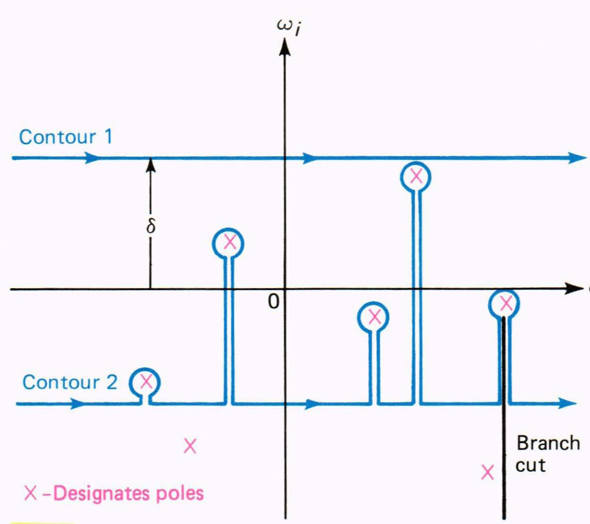

Figure 3—Integration contours in the complex frequency plane, to be used in the Laplace inversion integral appearing in Eq. 83. The poles and branch cuts are caused by both the dispersion function and the source function. In the long-time limit, the contour may be lowered far down in the negative ω_i plane.

The response of the fluid is governed by two classes of singularities: those resulting from the nature of the medium independent of initial and boundary conditions, and those imposed by boundaries, driving forces, and methods of excitation. To examine the intrinsic properties of the medium, it is highly useful to consider a fluid that has been excited by a delta function source long after the impulse has been applied. As Landau[4] has shown, in the asymptotic time limit, it is simpler to evaluate the frequency integral along a different line, denoted by Contour 2 in Fig. 3. The integrals from Contours 1 and 2 will be equivalent, provided that no singularities of the integrands are crossed during the deformation of the path. As Contour 1 is lowered into the negative ω_i half-plane, the contribution from its horizontal legs becomes vanishingly small as $t \rightarrow \infty$, leaving only the branch cuts and poles as contributors to the integral in the long-time limit. Since almost any physically realizable source, **S**, is likely to be free of branch cuts, it will suffice to consider only poles of the integrand. In this asymptotic time limit, most of the broad spectrum of transients generated by the impulse has been damped out by whatever loss mechanisms are represented by ω_i (e.g., eddy viscosity), leaving only the free waves characteristic of the fluid response, which oscillate at the eigenfrequencies. Then, in Eq. 83, the only contribution to the frequency integral comes from the poles of $\mathsf{D}^{-1}(\mathbf{k},\omega)$ or (exactly equivalent) from the zeros of the determinant of the D matrix, by Eq. 84. This condition, $\det(\mathsf{D}) = 0$, then establishes the wave propagation characteristics because it yields the generalized dispersion relation for the system.

Returning to Eq. 83 in the asymptotic time limit: for isolated singularities, the residue theorem allows the partial inverse frequency transform to be written as

$$\mathbf{U}(\mathbf{k},t) \xrightarrow[t \rightarrow \infty]{} \sum_{j}^{M} \exp(-i\omega_j t)\ [(\omega - \omega_j)\mathbf{U}(\mathbf{k},\omega)]_{\omega_j} \ ,$$

$$j = 1, 2, \ldots M \ , \qquad (85)$$

where the sum is over those M residues of D^{-1} that lie above Contour 2. The singularities, $\omega = \omega_j(\mathbf{k})$, are roots of the dispersion relation,

$$\det[\mathsf{D}(\mathbf{k},\omega)] = 0 = \prod_{j}^{M} [\omega - \omega_j(\mathbf{k})] \ , \qquad (86)$$

where the fundamental theorem of algebra has been used to write det(D) as a product of its factors. Each value of ω_j represents a different branch of the dispersion relation (acoustic, gravity, inertial, or Rossby waves), and each is an eigenfrequency of the fluid in the absence of boundaries, with a dispersion equation of the form

$$\omega = \omega_j(\mathbf{k}) \ . \qquad (87)$$

In the asymptotic time limit, $t \rightarrow \infty$, the least-damped oscillation (due to the uppermost pole in Fig. 3) persists longest; this may or may not be the lowest frequency wave.

Thus, for long times, the result of the Laplace inversion is given by Eq. 85, which may now be considered as the Fourier amplitude for the velocity field of the free waves. In many problems, it is easier to deal with individual Fourier components (i.e., Eq. 85) than with their sum (i.e., the Fourier inversion of Eq. 85).

EFFECTS OF BOUNDARIES ON THE DISPERSION RELATIONS

Having obtained the generalized dispersion equation for an infinite medium (Eq. 86), the effects of a finite depth of fluid will now be investigated. These primarily restrict the vertical wave number, m, to quantized values. Thus, horizontal boundaries have a waveguide effect on propagation in layered media. This constraint is expressed by additional equations involving m, which then must be used in conjunction with the infinite-medium dispersion relations above in solving for $\omega_j(\mathbf{k})$. The physical effect is to reduce propagation speeds of the wave modes in the fluid to values given by Eq. 59, in addition to quantizing the vertical wave number.

A Single-Layer Fluid

A single-layer fluid of constant depth, H, will be used to illustrate the effects of finite depth on wave propagation. Multilayer models having continuous density profiles composed of exponentially varying segments of the type given by Eq. 57 may be constructed from such layers; the Brunt-Väisälä frequency in each layer is constant but is discontinuous at the boundaries.

As is well known, the effect of depth is derived by imposing top and bottom boundary conditions.

The Bottom Boundary Condition. For a rigid bottom with no drag, the lower boundary condition is

$$\hat{z} \cdot \mathbf{u}_1 (\mathbf{x},t) = 0 \qquad \text{at } z = -H, \quad \text{all } t \ . \qquad (88)$$

This translates into a condition on the Fourier-Laplace transform of the Eckart field,

$$W(\mathbf{k},\omega) \exp(i\mathbf{k} \cdot \mathbf{x}) \ , \qquad (89)$$

which is satisfied for all values of z if the complex field is taken as

$$|W(\mathbf{k},\omega)| \exp\{i[m(z + H) - \pi/2]\} \ . \qquad (90)$$

The Surface Boundary Condition. The upper boundary condition at a free surface, taken as $z \approx 0$, is the dynamic condition for constant pressure, $dp/dt = 0$. From Eq. 31,

$$\frac{dp}{dt} = \frac{\partial p_1}{\partial t} + \mathbf{u}_0 \cdot \nabla p_1 + \mathbf{u}_1 \cdot \nabla p_0 = 0$$

$$\text{at } z = 0 \ , \qquad (91)$$

for which the partially transformed Eckart equivalent is

$$P(k,l,z = 0,\omega) = \left(\frac{g}{i\omega_d c}\right)\mathbf{U}(k,l,z = 0,\omega)\cdot\boldsymbol{\xi} \ . \quad (92)$$

Equation 92, involving an additional relationship between $\mathbf{U}$ and P at the surface, must be used in conjunction with the previous equations for those quantities in order to determine the vertical component m, which may now assume only those values that allow the pressure to vanish at the surface. To do this, six equations in six unknowns ($U, V, W, S, P,$ and n_z) must be solved simultaneously to obtain the dispersion equation for waves in the vertically bounded fluid.

We shall treat the case of steady waves with no mean flow in the asymptotic time limit and, in addition, continue to neglect Q. The slightly complicated algebra yields a dispersion equation

$$\omega^2(m) = \frac{N_0^2 + mg\tan mH}{1 + \tau\tan mH} \ , \quad (93)$$

where

$$N_0^2 \equiv N_z^2 + \Gamma_z g \quad (94)$$

is a transition frequency for surface waves. Equation 93 must be solved for the vertical wave number, m, which takes on an infinity of real, quantized values for body-wave modes, and a continuum of either real or imaginary values for the surface wave mode, with the transition occurring at $\omega^2 = N_0^2$. Thus,

$$m = m_\lambda(\omega,H,\tau), \quad \lambda = 0, \pm 1, \pm 2, \ldots, \quad (95)$$

where λ is the vertical mode index. Equation 93 is reminiscent of the dispersion equation for surface gravity waves, to which it reduces in the appropriate limit, but it is more general than that relation.

For the case of zero mean flow, Eqs. 86 and 93 are required in order to effect a solution for the dispersion equation. Thus, for $\mathbf{u}_0 = \mathbf{o}$, one obtains from Eq. 86, upon expansion of the determinant,

$$[(1 + i\tau)^2 - f_0^2/\omega^2][(1 + i\tau) - N_a^2/\omega^2 - (m_\lambda c/\omega)^2] - (1 + i\tau)[(1 + i\tau) - N_z^2/\omega^2](k^2 + l^2)c^2/\omega^2 = 0 \ . \quad (96)$$

The solutions are then

$$\omega = \omega_{j\lambda}(k,l,m,H); \quad j = 1, 2, \ldots, M; \quad \lambda = 0, \pm 1, \pm 2, \ldots . \quad (97)$$

Solutions to the Dispersion Equation

A number of familiar solutions will now be extracted from the dispersion function (Eq. 86) and the associated equation for the vertical wave number (Eq. 93). These will illustrate that the variety of waves claimed in the introduction is indeed contained in the formulation.

Uniform, Compressible, Nonrotating Fluid of Infinite Extent. In this case, $\mathbf{N}^2$, $\mathbf{u}_0$, f_0, g, and τ are zero and $H = \infty$. Then the condition $\det(\mathsf{D}) = 0$ is, upon expansion, equivalent to the dispersion equation,

$$\omega = \omega_j(\mathbf{k}) = \pm c\sqrt{k^2 + l^2 + m^2} \ , j = 1, 2 \ , \quad (98)$$

and only isotropic acoustic waves propagating toward and away from the origin of an infinite medium remain. The relation is analogous to the equation describing electromagnetic propagation in free space, with c playing the role of a limiting velocity of propagation in the fluid.

Acoustic-Gravity Waves in a Nonrotating, Streaming Fluid. If we neglect rotation and look at only high-frequency waves in a streaming, compressible, stratified medium of semi-infinite vertical extent, Eq. 86 yields the dispersion equation for hybrid acoustic-internal oscillations. One obtains an implicit dispersion equation,

$$\omega_d^2(1 + i\tau)\{\omega_d^2(1 + i\tau)[\omega_d^2(1 + i\tau) - (k^2 + l^2 + m^2)c^2 - N_a^2] + N_z^2(k^2 + l^2)c^2\} = 0 \ , \quad (99)$$

which has three modes, two of which propagate in opposite directions. The quantity in the braces describes coupled acoustic-internal gravity waves in this medium, which, in the high-frequency, loss-free limit, reduce to the case of an acoustic wave in a stratified flow:

$$\omega = \omega_j(\mathbf{k}) \approx \mathbf{u}_0\cdot\mathbf{k} \pm \sqrt{N_a^2 + (\mathbf{k}c)^2} \ , \quad j = 1, 2 \ . \quad (100)$$

The atmospheric cutoff frequency, N_a, is given by Eq. 25. Equation 100 describes, among other things, the propagation of infrasonic waves in the atmosphere.[5]

The remaining mode, given by

$$\omega_d^2(1 + i\tau) = 0 = (\omega - \mathbf{u}_0\cdot\mathbf{k})\{\omega - \mathbf{u}_0\cdot\mathbf{k} + i[A_h(k^2 + l^2) + A_v(m^2 + \Gamma_0^2)]\} \ , \quad (101)$$

simply represents advection by the stream of a perturbation of scale $2\pi/|\mathbf{k}|$, which, when observed in a fixed coordinate system, appears as a damped wave, the real part of whose frequency is ω.

Internal Gravity Waves in a Rotating Medium. The internal wave dispersion relation is obtained from Eq. 96 in the limit of an incompressible fluid by allowing $c^2 \rightarrow \infty$ and using the relationship $N_a/c = -\Gamma_0$. Setting $\mathbf{u}_0$ and τ equal to zero gives

$$\omega_j(\mathbf{k}) = \pm \sqrt{\frac{N_z^2\ (k^2 + l^2) + f_0^2\ (m^2 + \Gamma_0^2)}{k^2 + l^2 + m^2 + \Gamma_0^2}}\ , \tag{102}$$

where the non-Boussinesq term, Γ_0, is usually neglected. In the long wavelength limit, $k^2 + l^2 = 0$ and $\omega_{\text{low}}^2 = f_0^2$, while in the short wavelength limit, $m^2 + \Gamma_0^2 = 0$ (corresponding to purely vertical propagation) and $\omega_{\text{hi}}^2 = N_z^2$.

If f_0 is negligible, another familiar form of internal wave dispersion equation is obtained:

$$\omega_j^2(\theta) = N_z^2 \left(\frac{k^2 + l^2}{k^2 + l^2 + m^2} \right) = N_z^2 \sin^2 \theta\ , \tag{103}$$

where θ is the propagation angle with respect to the vertical.

Surface Gravity Waves on a Rotating, Stratified, and Bounded Fluid. We will treat the surface gravity wave for the case of a nonstreaming, dissipation-free, stratified medium on a rotating plane. A rewriting of Eq. 86 for this case yields

$$(1 - f_0^2/\omega^2)[1 - N_a^2/\omega^2 - (mc/\omega)^2]$$
$$- (1 - N_z^2/\omega^2)\ (k^2 + l^2)(c^2/\omega^2) = 0\ . \tag{104}$$

We next solve this equation for m and reinterpret that quantity as a vertical attenuation coefficient, μ, via

$$m = \pm i\mu \tag{105}$$

in order to take into account the essentially evanescent, edge-wave nature of surface waves. This reciprocal e folding distance is

$$\mu = \sqrt{[(k^2 + l^2)(\omega^2 - N_z^2)/(\omega^2 - f_0^2)] - [(\omega^2 - N_a^2)/c^2]}$$
$$\xrightarrow[c^2 \to \infty]{} \sqrt{\Gamma_0^2 + [(k^2 + l^2)(\omega^2 - N_z^2)]/(\omega^2 - f_0^2)}\ , \tag{106}$$

where Eq. 56 has been used for N_a/c. This quantity reduces to the familiar expression for the vertical wave number in a nonrotating Boussinesq fluid when $f_0 = \Gamma_0 = l = 0$:

$$\mu^2 = k^2(1 - N_z^2/\omega^2)\ . \tag{107}$$

With the interpretation of μ given by Eq. 106, Eq. 93, as derived from the upper and lower boundary conditions, becomes

$$\omega^2 = N_0^2 + \mu g \tanh \mu H\ , \tag{108}$$

where the transition frequency, N_0 (cf. Eq. 94), is so named because it is at this frequency that waves obeying Eq. 93 make the transition from imaginary to real vertical wave numbers and change character from the surface to the lowest mode internal gravity waves.[2] If the medium is neither rotating nor stratified, the usual expressions for surface gravity waves are recovered from Eqs. 106 and 108:

$$\omega_j(\mathbf{k}) = \pm \sqrt{g \sqrt{k^2 + l^2}\ \tanh \sqrt{k^2 + l^2})\ H}, \tag{109}$$

which, for deep water, approaches

$$\xrightarrow[\mu H \to \infty]{} \pm \sqrt{g \sqrt{k^2 + l^2}}\ , \tag{110}$$

whereas for shallow water, the limit is

$$\xrightarrow[\mu H \to 0]{} \pm \sqrt{gH(k^2 + l^2)}\ . \tag{111}$$

Equations 106 and 108 also show that the limiting velocity for long wavelength surface waves in an unstratified fluid is $\sqrt{gH}$ if the fluid is shallow, and c if it is deep. If $\Gamma_0 = N_a = N_z = f_0 = 0$, but $c \neq \infty$, Eq. 106 yields

$$\mu = \sqrt{k^2 + l^2 - \omega^2/c^2}\ . \tag{112}$$

Upon substitution into Eq. 108, this gives, in the long-wave limit, a phase speed, c_ϕ, of

$$c_\phi^2 = \frac{\omega^2}{k^2 + l^2} = \frac{1}{(1/c^2) + (1/gH)}\ , \tag{113}$$

which approaches

$$c_\phi^2 \to gH, \qquad gH \ll c^2\ . \tag{114}$$

In a deep fluid, but one still shallow compared with a wavelength,

$$c_\phi^2 \to c^2,\ gH \gg c^2\ . \tag{115}$$

Inertial Oscillations. Near the inertial frequency, $\omega^2 \approx f_0^2$, and from Eq. 96,

$$k^2 + l^2 \approx 0\ . \tag{116}$$

Thus, inertial oscillations are essentially very long-length waves of complex frequency:

$$(\omega_r + i\omega_i)(1 + i\tau) = \pm f_0\ , \tag{117}$$

or, from Eq. 101 for this case,

$$\omega_r = \pm f_0(1 + \tau^2) \tag{118}$$

$$\omega_i = \mp f_0 \tau/(1 + \tau^2)$$
$$= \mp f_0 m^2 A_v/[1 + (m^2 A_v/\omega_r)^2]\ , \tag{119}$$

where we have neglected Γ_0 compared with m. The real part of the frequency is shifted somewhat from f_0 by the viscosity. Also, the damping decrement for inertial oscillations is proportional to the vertical wave number and vertical eddy viscosity, indicating that primarily upward or downward propagation is dominant.

This will be discussed in the paragraphs under the heading, A Parameter Space for the Atmosphere.

Subinertial Waves. In the absence of the beta effect, subinertial waves can only occur in a fluid for which $N_z^2 < f_0^2$, i.e., a rapidly spinning or weakly stratified system. In the incompressible limit and at frequencies well below the inertial frequency, Eq. 96 gives a dispersion equation for these waves.

$$\omega^2 = N_z^2 + f_0^2(m^2 + \Gamma_0^2)/(k^2 + l^2) . \quad (120)$$

If θ is the angle of propagation measured from the vertical, this may be rewritten as

$$\omega = \omega_j = \pm \sqrt{N_z^2 + f_0^2 \cot^2 \theta} , \quad (121)$$

illustrating that subinertial waves, as with inertial oscillations, propagate mainly vertically.

Thus, as shown by subsections 1 through 6, Eqs. 86 and 93 describe without inordinate complication a very wide range of waves in the rotating, stratified, single-layer fluid. The general statement, Eq. 86, is a fourth-order polynomial describing left and right propagation of two acoustic and two internal gravity waves. The introduction of lower and upper boundaries adds two additional surface-wave modes. In addition, the vertical eigenfrequency structure is superimposed on the body-wave modes for the case of the layered fluid.

A PARAMETER-SPACE DIAGRAM

The Fluid Parameters

We will now develop an analytical and graphical technique for sorting out and classifying the variety of modes in the model fluid. This technique will also illustrate the behavior of the vector index of refraction in a multidimensional parameter space. In the absence of a mean flow and sources, the normalized parameters in the problem are N_z^2/ω^2, N_0^2/ω^2, N_a^2/ω^2, f_0^2/ω^2, $\omega A/c^2$, and mH. It is obviously impossible to represent graphically more than three of these at once, and it is even difficult to illustrate clearly the behavior of more than two. However, the parameters that differentiate the fluid most vividly, for example, are stratification and Coriolis force. Therefore, the two fundamental parameters will be taken as proportional to N_z and f_0.

The buoyancy parameter:

$$B_z^2 = N_z^2/\omega^2 .$$

The Coriolis parameter:

$$F_0^2 = f_0^2/\omega^2 .$$

Solution for the Index of Refraction

Referring to Fig. 2, when $\mathbf{u}_0 = \mathbf{0}$, the components of the vector index of refraction along the coordinate axes are

$$kc/\omega = n \sin\theta \cos\theta , \quad (122)$$

$$lc/\omega = n \sin\theta \sin\theta , \quad (123)$$

$$mc/\omega = n \cos\theta , \quad (124)$$

where in the present notation,

$$n^2 = (k^2 + l^2 + m^2)c^2/\omega^2 . \quad (125)$$

We choose to deal with Eq. 96 with $\tau = 0$ for simplicity. Then, with the index components above substituted in Eq. 85 or 96, one obtains

$$(1 - F_0^2)(1 - B_a^2 - n^2 \cos^2\theta) - (1 - B_z^2)(n^2 \sin^2\theta) = 0 . \quad (126)$$

This can be factored into the form

$$n(\theta) = \sqrt{\frac{1 - B_a^2}{1 - (B_z^2 - F_0^2)(\sin^2\theta)/(1 - F_0^2)}} . \quad (127)$$

The index of refraction is a function of the polar angle and the stratification and Coriolis parameters. Since n is defined as the ratio of the speed of sound to the phase speed of the wave, $1/n$ is proportional to the phase speed, $c_\phi = \omega/\sqrt{k^2 + l^2 + m^2}$, at different values of θ. Therefore, a three-dimensional surface in (x,y,z) space may be defined by the locus of the tip of a vector whose direction is that of the wave vector and whose length is proportional to the phase speed, or to $1/n(\theta)$. This figure is called the *phase velocity* or the *wave normal surface* and is the reciprocal to the more familiar wave number surface, $n(\theta)$.[6]

Figure 4 shows a two-dimensional curve generated by the intersection of a plane containing the z axis with the phase velocity surface. Such surfaces typically have either lemniscate or spheroidal shapes. The figure illustrates the behavior of the wave number surface, $1/n(\theta)$, and its reciprocal, $n(\theta)$; the normal to the latter defines the direction of the group velocity $\mathbf{c}_g = \nabla_\mathbf{k}\omega(\mathbf{k})$. The dumbbell-shaped lemniscate also varies as F_0^2 and B_z^2 vary, of course. By studying these surfaces along with

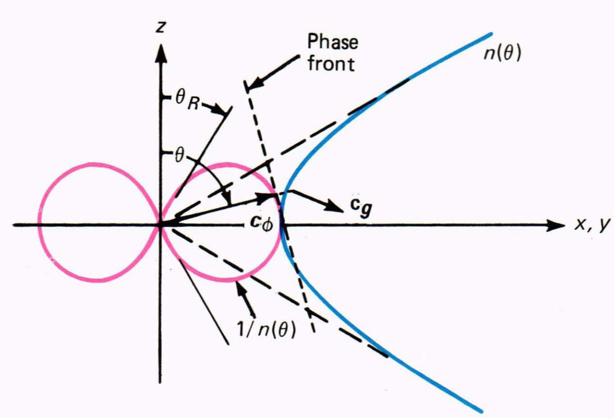

Figure 4—Phase velocity surface, $1/n(\theta)$, and wave number surface, $n(\theta)$. The former gives the relative variation in phase velocity c_ϕ as the angle from the vertical, θ, changes. The direction of the group velocity, $\mathbf{c}_g$, is specified by the normal to the wave number surface.

certain other critical surfaces in the parameter space, one can understand something of the complicated behavior of n by examining allowed and forbidden regions of propagation in that space.

A Parameter Space for the Atmosphere

As a relatively simple example of a parameter space that contains the basic features of a semi-infinite geophysical fluid, consider the two-dimensional diagram shown in Fig. 5. This figure is constructed for an atmosphere with a plane lower boundary. The vertical axis is the buoyancy parameter, N_z^2/ω^2, and the horizontal axis is the Coriolis parameter, f_0^2/ω^2. Such a graph is analogous to a diagram for magnetized plasma given by P. C. Clemmow, R. F. Mullaly, and W. P. Allis.[7]

If one now considers a fluid whose stratification and rotation are slowly varying throughout its volume, the behavior of a wave in the fluid can be deduced partially from the behavior of the phase velocity surface in the parameter space. Equivalently, for fixed N_z and f_0, as the frequency of the wave is made to vary, its location in parameter space moves; high frequencies are found in the lower left-hand corner and low frequencies in the upper right-hand corner.

The small figures at various locations on the diagram are phase velocity surfaces, as in Fig. 4, with the isotropic velocity of sound shown by dotted circles. One may think of this diagram itself as a kind of parameter "fluid," with the stratification increasing with height and the rotation rate increasing to the right. Furthermore, "up" in the parameter fluid corresponds to "up" in the real fluid. At various points in the "fluid," we may think of small steady-state wave-making devices that radiate waves; the wave normal surfaces are surfaces of constant phase about each device, and the radius vector is proportional to the distance traveled by the wave at a given angle in physical space. The topological character of these phase velocity surfaces changes as the wavemakers cross the critical parameter surfaces mentioned above.

To see how this comes about, consider Eq. 127 for $n(\theta)$. The index of refraction vanishes along a line in $B_z^2 - F_0^2$ space given by $B_a^2 = 1$. (In Fig. 5, B_a^2 has been chosen equal to 0.81 B_z^2, corresponding to the atmospheric cutoff case for the earth.) It also vanishes for $F_0^2 = 1$, unless $\theta \equiv 0$. The zeros of $n(\theta)$ are called *cutoffs* and form one set of bounding critical surfaces in the parameter space. On one side of the boundary, the radical in Eq. 127 is negative, n is imaginary, and no propagation is possible; the region is called *evanescent*, and a wave crossing the boundary will decay in real space exponentially. On the other side, propagation is possible and the wave oscillates sinusoidally.

The other critical surfaces are generated by *resonances*, or regions where $n^2 = \infty$ in the lossless case (in the case with viscosity present, n^2 is large but finite). From Eq. 127, this condition obtains at a resonance angle, θ_R, and resonance frequency, ω_R, given by

$$\omega_R^2 = f_0^2 \cos^2 \theta_R + N_z^2 \sin^2 \theta_R \; . \qquad (128)$$

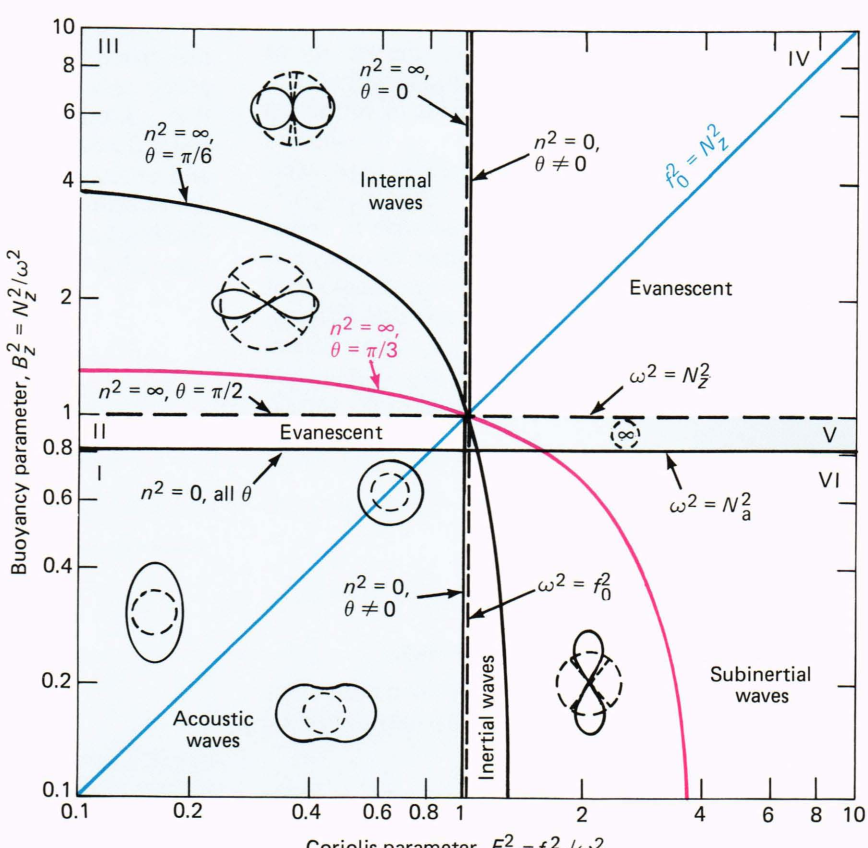

Figure 5—Two-dimensional parameter "fluid" for a stratified, rotating, compressible atmosphere. The diagram is divided into six regions as specified by roman numerals, with the boundaries set (a) by resonances, where the index of refraction is infinite ($n^2 = \infty$), and (b) by cutoffs, where $n^2 = 0$. The locations of resonances depend on θ and are shown for $\theta_R = 0$, $\pi/6$, $\pi/3$, and $\pi/2$, while the locations of cutoffs are independent of angle. No wave propagation occurs in evanescent regions. In the propagating regions, the phase velocity varies with angle as shown by the small inset figures, which are phase velocity surfaces as on Fig. 4. The isotropic velocity of sound is indicated by the dotted circles and the resonance cones by the dotted lines passing through their centers. Various types of fluid waves exist in the regions as noted.

At $n^2 = \infty$, one has $c_\phi = 0$, and no propagation is possible; once again, n is real on one side of the critical surface and imaginary on the other. The angle θ_R is the angle at which c_ϕ becomes zero (Fig. 4), thereby limiting the angular width of the lemniscates shown in the parameter fluid; propagation cannot occur at angles in physical space outside the region defined by the resonance cone $\theta = \theta_R$.

The combination of resonances and cutoffs divides the B_z^2 - F_0^2 parameter space into six regions on Fig. 5, numbered clockwise from I to VI. As each of the critical bounding surfaces is crossed, the topological character of the wave normal surfaces changes; hence, so do the propagation characteristics. While the cutoffs are independent of angle, the resonances depend on θ via Eq. 128. Figure 5 shows how these infinities vary with B_z^2 and F_0^2 for constant values of θ equal to 0, 30, 60, and 90 degrees.

The B_z^2 - F_0^2 diagram of Fig. 5 gives a vantage point in locating where the various fluid modes may exist in the "fluid." The high-frequency acoustic or acoustic-gravity waves are obviously confined to the lower left-hand corner of Region I (cf. Eq. 99); their phase velocities all exceed c (in the semi-infinite case) and are nearly isotropic; in fact, they are exactly isotropic along the line $f_0^2 = N_z^2$. Internal waves, which are constrained to the frequency interval

$$f_0^2 < \omega^2 < N_z^2 , \tag{129}$$

must therefore be located in Region III and are characterized by near-horizontal propagation. Regions I and III are separated by the small evanescent Region II, which is located between the Brunt-Väisälä and atmospheric-cutoff frequencies.

In Region IV, where both f_0^2 and N_z^2 are greater than ω^2, waves are also evanescent. Regions V and VI, which correspond to a rapidly rotating, weakly stratified fluid, are not generally accessible in geophysical fluids but may readily be so in planetary fluids or laboratory experiments. They allow propagation of sub-inertial waves mainly in the vertical, as the phase velocity surface suggests (cf. Eq. 121). In Region V, propagation is possible at near-horizontal angles and at frequencies near f_0. Inertial oscillations occur near the boundary between Regions VI and I, at $F_0^2 \approx 1$.

SUMMARY AND CONCLUSIONS

A model is presented for linear waves propagating in a compressible, uniformly stratified, viscous fluid flowing with a constant mean velocity on a β plane. The linearized Navier-Stokes equations are used in conjunction with the isentropic form of the First and Second Laws of Thermodynamics and equations of state and internal energy to construct a linear response theory for free and forced waves propagating in the fluid. The complete, formal solution for the first-order velocity field is derived in terms of Fourier-Laplace inversion integrals, given by Eq. 83. The behavior of the field in time and space depends on two separate features of the solution. First, the intrinsic properties of the fluid (stratification, rotation, sound speed, viscosity, and streaming velocity) establish a response that is independent of initial and boundary conditions; the response is described by the general dispersion tensor, $\mathsf{D}^{-1}(\mathbf{k},\omega)$. These functions summarize all of the wave propagation characteristics that are independent of boundaries or forcing but which are governed by the nature of the medium itself. For a layered fluid, they yield a general dispersion relation (Eqs. 86 and 93 combined), whose solutions for frequency as a function of wave vector determine the behavior of planetary, inertial, internal, and acoustic waves and the various hybrid combinations that result from coupling between these types. Second, the extrinsic features of the medium (the presence of boundaries, initial values of the fluid variables, or forcing applied by external means) then modify the intrinsic response in a variety of ways. Mathematically, these features are given by a source function $\mathbf{S}$ that specifies mixed initial and boundary values. Boundaries can also be introduced directly through boundary conditions, and an example for a stratified, rotating, single-layer ocean is given. The top-bottom wave guide effect given by Eq. 93 delimits the vertical wave number and, when used in the infinite-medium dispersion relation, recovers acoustic, internal, and surface gravity waves in recognizable forms (Eqs. 98, 103, and 109).

Finally, the parameter "fluid" is introduced as a means of classifying waves and showing their allowed and forbidden regions of propagation as functions of the fluid parameters. Cutoffs and resonances in the index of refraction occur at critical values of the parameters for stratification and Coriolis force, and these appear as bounding surfaces in the "fluid." An example is given for a parameter "atmosphere" in Fig. 5.

The theory is capable of addressing a broad class of problems in atmospheres, oceans, and laboratory fluids; some examples, in addition to propagation of simple free waves, are waves in channeled flows, wave coupling to form hybrid types, oceanic response to wind and barometric forcing, and baroclinic and barotropic instabilities.

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GLOSSARY OF SYMBOLS

(Minor subscripts and symbols have been omitted.)

A, tensor eddy viscosity
A_h, horizontal elements of viscosity
A_v, vertical element of viscosity
A, Eckart specific volume field
a, coefficient of thermal expansion
$\mathbf{B}^2 = (B_x^2, B_y^2, B_z^2)$, baroclinic/buoyancy parameter
B_a^2, buoyancy parameter for cutoff frequency
c, speed of sound
C_α, specific heat at constant volume
C_p, specific heat at constant pressure
c_e, effective propagation speed
c_ϕ, wave phase speed
$\mathbf{c}_g$, wave group velocity
D, tensor dispersion function
D^{-1}, inverse of dispersion function
D_{ij}, i-jth matrix element of D
cof(D), cofactor matrix of D
det(D), determinant of D
e, internal energy per unit mass
$\mathbf{F}_0 = \mathbf{\Omega}/\omega_d$, normalized Coriolis vector
$F_0^2 = f_0^2/\omega_d^2$, Coriolis parameter
$f = f_0 + \beta y$, variable Coriolis frequency
f_0, constant Coriolis frequency
$\mathfrak{F} = \mathfrak{F}_x \mathfrak{F}_y \mathfrak{F}_z$, Fourier transform operators
$\mathbf{g} = -g\hat{z}$, acceleration of gravity
g, scalar acceleration of gravity
H, layer thickness
I, unit tensor
i, j, tensor indices for x, y, and z
j, index for roots of dispersion equation
K, tensor eddy heat diffusivity
K_h, horizontal elements of diffusivity
K_v, vertical element of diffusivity
$\mathbf{k} = (k, l, m)$, wave vector
k, x-component of wave vector
k_h, horizontal component of wave vector
$\mathfrak{L}$, Laplace transform operator
l, y-component of wave vector
m, z-component of wave vector
$\mathbf{N}^2 = (N_x^2, N_y^2, N_z^2)$, vector frequency parameter
$N_a = -c\rho_0'/2\rho_0$, atmospheric cutoff frequency
$N_x^2 = fgv_0/c^2$, x-baroclinic frequency
$N_y^2 = -fgu_0/c^2$, y-baroclinic frequency
$N_z^2 = -g(\rho_0'/\rho_0 + g/c^2)$, Brunt-Väisälä frequency
$\mathbf{n} = (n_x, n_y, n_z)$, vector index of refraction
$n = |\mathbf{n}|$, absolute value of $\mathbf{n}$
$\mathbf{n}^+$, adjoint vector index of refraction
$\mathbf{n}_0 = (n_{0x}, n_{0y}, n_{0z})$, vector index for zero Coriolis force
$\mathbf{n}_0^*$, complex conjugate of $\mathbf{n}_0$
$\mathbf{n}_0^+$, adjoint vector index of refraction
P, Eckart pressure field
p, pressure
p_0, zero-order pressure
p_1, first-order pressure
Q, Eckart heating rate field
q, heat per unit mass
q_0, zero-order heat per unit mass
q_1, first-order heat per unit mass
$\dot{q}_0 = dq_0/dt$, zero-order heating rate
$\dot{q}_1 = dq_1/dt$, first-order heating rate
R, Eckart density field
$\hat{r}$, unit vector in radial direction
$\mathbf{s}$, wave slowness
S, Eckart entropy field
s, entropy per unit mass
s_0, zero-order entropy per unit mass
s_1, first-order entropy per unit mass
$\mathbf{S}$, vector source function
T, temperature
T', Eckart temperature field
T_0, zero-order temperature
T_1, first-order temperature
t, time
$\mathbf{U} = (U, V, W)$, Eckart velocity field
U, x-component of velocity field
$\mathbf{u} = (u, v, w)$, fluid velocity
$\mathbf{u}_0 = (u_0, v_0, 0)$, zero-order velocity
$\mathbf{u}_1 = (u_1, v_1, w_1)$, first-order velocity
u, x-component of velocity
u_0, x-component of zero-order velocity
u_1, x-component of first-order velocity
V, y-component of Eckart velocity field
v, y-component of velocity
v_0, y-component of zero-order velocity
v_1, y-component of first-order velocity
W, z-component of Eckart velocity field
w, z-component of velocity
w_0, z-component of zero-order velocity
w_1, z-component of first-order velocity
$\mathbf{x} = (x, y, z)$, Cartesian coordinates (x, east; y, north; z, up)
$\hat{x}$, unit vector in x-direction
$\hat{y}$, unit vector in y-direction
$\mathbf{Z}$, generalized vector wave impedance
Z_0, acoustic impedance
$\hat{z}$, unit vector in z-direction
z_0, reference level
α, specific volume
α_0, zero-order specific volume
α_1, first-order specific volume
β, beta-plane parameter
$\mathbf{\Gamma} = (\Gamma_x, \Gamma_y, \Gamma_z)$, attenuation vector
$\Gamma_g = -g/c^2$, compressibility reciprocal scale height
$\Gamma_0 = \rho_0'/2\rho_0$, transition attenuation coefficient
$\Gamma_x = -fv_0/c^2$, x-baroclinic attenuation coefficient
$\Gamma_y = fu_0/c^2$, y-baroclinic attenuation coefficient
$\Gamma_z = \rho_0'/2\rho_0 + g/c^2$, vertical attenuation coefficient
γ, ratio of specific heats
δ, imaginary coordinate for Laplace inversion
δ_{ij}, Kronecker index
ϵ_{ijk}, permutation index
η, elevation of fluid surface above equilibrium
θ, polar angle between z-axis and wave vector
θ_R, resonance polar angle
κ, heat diffusivity parameter
Λ, latitude
λ, vertical mode index
μ, imaginary vertical wave number
$\xi_x = fv_0/g$, x-slope of isopycnal surface
$\xi_y = -fu_0/g$, y-slope of isopycnal surface
ξg, reduced gravity vector
ρ, density
ρ_0, zero-order density
ρ_1, first-order density
τ, eddy diffusivity parameter
ϕ, azimuth angle between x-axis and horizontal component of wave vector
ψ, arbitrary scalar variable
$\mathbf{\Omega}$, Coriolis vector
Ω_E, angular speed of planet
ω, radian frequency
ω_d, Doppler-shifted frequency
ω_i, imaginary part of frequency
ω_j, frequency roots to dispersion equation
ω_R, resonance frequency
ω_r, real part of frequency

THE AUTHOR

JOHN R. APEL is a physicist/oceanographer and chief scientist of the Milton S. Eisenhower Research Center. He holds degrees in physics and mathematics from the University of Maryland, has a Ph.D in applied physics from The Johns Hopkins University, and has published extensively in plasma physics and oceanography. Formerly the director of the Pacific Marine Environmental Laboratory of the National Oceanic and Atmospheric Administration, he is the recipient of several federal awards, including a U.S. Commerce Department gold medal for work on the ocean-viewing spacecraft, Seasat. Dr. Apel has been an adjunct professor at the Universities of Miami and Washington and is currently lecturing in physical oceanography and atmospheric science at the Whiting School of Engineering at Johns Hopkins. He expects to publish a graduate-level textbook on principles of ocean physics in the fall of 1986. The theoretical work on waves in geophysical fluids, described in this issue, stems from his attempts to provide a unified view of the wide range of waves that exist in the ocean or the atmosphere.