a METHOD of ANALYSIS for CLAMPED-FREE CYLINDRICAL SHELLS

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Thin-walled cylindrical shells in missile structures are subjected during flight to non-axial symmetric load and temperature distributions. Deflection analysis under such loads, in the case of ramjet- and turbojet-powered vehicles, is complicated by clamped-free boundary conditions and non-uniform shell thickness of such engine structures. This paper discusses a method of deflection analysis for a clamped-free cylinder when loads and temperatures are symmetrically distributed about planes passing through the axis of the cylinder and for shell thickness variations along its axis.

Thin-walled shells have found widespread use in aerospace, naval, and pressure-vessel structures because of their high strength-to-weight ratios and ease of fabrication. In flight vehicles particularly, where minimum weight is of utmost importance, large portions of the airframes are of this type of construction. Engine and body structures, for instance, are usually shells of revolution. Unlike most pressure vessels where load and temperature distributions are uniform or rotationally symmetric, flight structures are frequently subjected to loads and temperatures that vary in both the circumferential and longitudinal directions. Such a situation arises in a missile flying at an angle of attack.

In ramjet-powered missiles in particular, it is necessary to be able to predict accurately the deflections of the lip of the inlet since the performance of the engine is strongly influenced by small changes in the inlet geometry. In addition to the complications arising from the lack of axial symmetry of the loads and temperatures, the wall thickness often varies in the longitudinal direction. While there is an extensive literature on shells of revolution of constant thickness subjected to uniform or simple forms of rotationally symmetric loading, relatively little has been published on unsymmetric loadings. The case with clamped-free* boundary conditions, which occurs in the inlet and tailpipe of ramjet- and turbojet-powered vehicles, does not appear to have been treated at all.

In this paper we summarize a method of deflection analysis for clamped-free cylinders under the action of load and temperature distributions that are symmetrical about a plane passing through the axis of the cylinder. This is drawn from a more detailed development of the theory and description of the analytical procedure that has been programmed for the IBM 7094 computer.[1]

Method of Analysis

The geometric notation and coordinate system for the clamped-free circular cylinder with a longitudinally varying wall thickness is shown in Fig. 1. It is assumed that both the loading and temperature distribution are symmetrical about a plane passing through the axis of the cylinder and that θ is measured from this plane. The case in which more than one plane of symmetry exists is treated in Ref. 1. Closed-form solutions for the

* Clamped at one end and free at the other.

[1] R. M. Rivello and T. M. Rankin, *Analysis of Clamped-Free Cylindrical Shells Under the Action of Nonaxisymmetric Loads and Temperatures,* The Johns Hopkins University, Applied Physics Laboratory, CF-3021, Mar. 8, 1963.

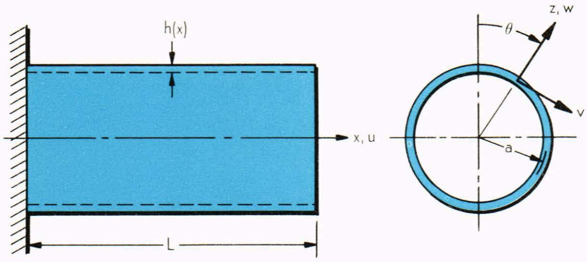

Fig. 1.—Geometry and coordinates for a circular cylinder.

deflections are not possible because of the arbitrary nature of the load, temperature, and thickness variations; it is therefore necessary to resort to approximate methods.

The analysis makes use of variational methods. In particular, it is based upon the Rayleigh-Ritz method used in conjunction with the principle of the stationary value of the total potential.[2] In this method the total potential energy (the sum of the strain energy of the structure and the potential energy of the applied forces) is expressed in terms of the displacements of the structure. These displacements are, in turn, related to a set of generalized coordinates. The magnitudes of the generalized coordinates are found from the condition that, for equilibrium to exist, the total potential energy must have a stationary value for any arbitrary variation of the generalized coordinates.

For the case at hand, the strain energy of the shell may be written as

$$U = U_F + U_T, \tag{1}$$

where U_F is the energy associated with the strain due to the applied forces[3]

$$\begin{aligned} U_F = \frac{1}{2}\int_0^L \int_0^{2\pi} \frac{Eh}{(1-v^2)} & \left[(\epsilon_1+\epsilon_2)^2 - 2(1-v)\left(\epsilon_1\epsilon_2 - \frac{\omega^2}{4}\right)\right] ad\theta\, dx \\ + \frac{1}{2}\int_0^L \int_0^{2\pi} \frac{Eh^3}{12(1-v^2)} & \Bigg[(K_1+K_2)^2 - 2(1-v)(K_1K_2 - \tau^2) + \frac{2}{a}(\epsilon_1 K_1 - \epsilon_2 K_2) \\ & - (1-v)\frac{\omega\tau}{a} + \frac{\epsilon_2^2}{a^2} + (1-v)\frac{\omega^2}{2a^2}\Bigg] ad\theta\, dx, \end{aligned} \tag{2}$$

and U_T, the strain energy resulting from the thermal strain, is

$$U_T = -\int_0^L \int_0^{2\pi} \left[N_T(\epsilon_1+\epsilon_2) + M_T(K_1+K_2) - \frac{\alpha^2 E}{1-v}\int_{-\frac{h}{2}}^{\frac{h}{2}} T^2\, dz \right] ad\theta\, dx. \tag{3}$$

In these equations, E is the modulus of elasticity and v is Poisson's ratio of the material. Subscripts 1 and 2 refer to the longitudinal and tangential directions. For a thin shell,

$$N_T(x,\theta) = \frac{\alpha E}{1-v}\int_{-\frac{h}{2}}^{\frac{h}{2}} T(x,\theta,z)\, dz,$$

and

$$M_T(x,\theta) = \frac{\alpha E}{1-v}\int_{-\frac{h}{2}}^{\frac{h}{2}} T(x,\theta,z)\, zdz, \tag{4}$$

where α is the coefficient of expansion and $T(x, \theta, z)$ is the change in temperature from its initial value.

The normal strains ϵ_1 and ϵ_2 and the shearing strain ω are related to the displacements u, v, and w of the middle surface of the shell (Fig. 1) by the relationships[3]

$$\epsilon_1 = \frac{\partial u}{\partial x}, \qquad \epsilon_2 = \frac{1}{a}\frac{\partial v}{\partial \theta} + \frac{w}{a},$$

and

$$\omega = \frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial \theta}. \tag{5}$$

Similarly, the curvatures K_1 and K_2, and the twist τ of the middle surface are given by[3]

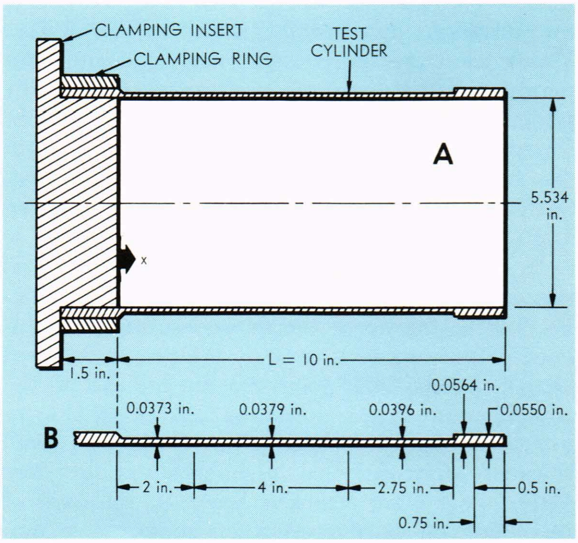

Fig. 2.—Test cylinder dimensions, showing (A) overall dimensions of the cylinder, and (B) the average wall thicknesses of particular segments.

[2] N. J. Hoff, *The Analysis of Structures*, John Wiley and Sons, Inc., New York, 1956.

[3] V. V. Novozhilov, *The Theory of Thin Shells*, P. Noordhoff, Ltd., Grominger, The Netherlands, 1959.

$$K_1 = -\frac{\partial^2 w}{\partial x^2}, \quad K_2 = -\frac{1}{a^2}\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta}\right),$$

and

$$\tau = -\frac{1}{a}\left(\frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x}\right). \tag{6}$$

By substituting Eqs. (2) and (3) into Eq. (1) and then using Eqs. (5) and (6), the strain energy may be expressed in terms of the displacement functions u, v, and w. The potential energy of the applied forces may also be written in terms of these quantities. For a distributed loading having components with intensities p_1, p_2, and p_z in the directions x, θ, and z, the potential energy is

$$V = -\int_0^L \int_0^{2\pi} (p_1 u + p_2 v + p_z w) a d\theta \, dx. \tag{7}$$

The potential for concentrated forces is

$$V = -\sum_i (P_{1,i} u_i + P_{2,i} v_i + P_{z,i} w_i), \tag{8}$$

where $P_{1,i}$, $P_{2,i}$, and $P_{z,i}$ are the components of the ith force and u_i, v_i, and w_i are the components of the displacement at the point of application of the force. If both distributed and concentrated loads act simultaneously, the potential of the forces is the sum of Eqs. (7) and (8). The total potential of the system is obtained by adding U and V.

In the Rayleigh-Ritz method the displacements are approximated by a series, each term of which must satisfy the displacement or essential boundary conditions, which in this case are

$$u = v = w = \frac{\partial w}{\partial x} = 0 \quad \text{at } x = 0. \tag{9}$$

In addition, for the particular case under discussion, the deflections must be symmetrical about the planes of loading and thermal symmetry since the

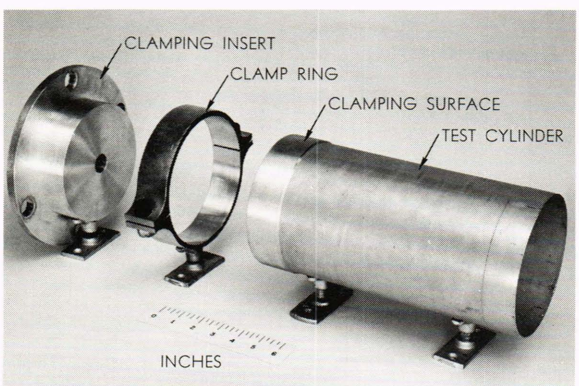

Fig. 3.—Test cylinder, with the support for the clamped end.

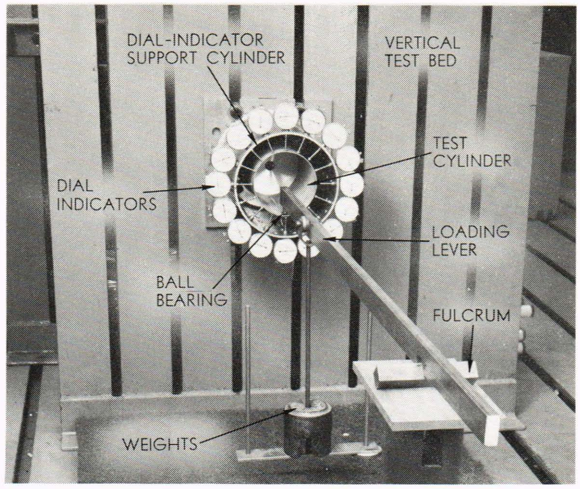

Fig. 4.—Laboratory setup for testing a clamped-free cylinder.

structure itself is axisymmetric. Series which satisfy the above conditions are

$$u = \sum_{m=1}^{N} \sum_{n=0}^{N} a_{mn} \left(\frac{x}{L}\right)^m \cos(n\theta),$$

$$v = \sum_{m=1}^{N} \sum_{n=1}^{N} b_{mn} \left(\frac{x}{L}\right)^m \sin(n\theta), \tag{10}$$

and

$$w = \sum_{m=2}^{N+1} \sum_{n=0}^{N} c_{mn} \left(\frac{x}{L}\right)^m \cos(n\theta),$$

if θ is measured from this symmetry plane. The total potential $(U + V)$ may be expressed in terms of the coefficients a_{mn}, b_{mn} and c_{mn} by means of Eqs. (10). These coefficients can be considered to be generalized coordinates of the system since they prescribe the deflections. From the principle of the stationary value of the total potential, $(U + V)$ must not change during any virtual displacement. If the virtual displacements are obtained by taking variations in a_{mn}, b_{mn}, and c_{mn}, this requires that

$$\frac{\partial(U+V)}{\partial a_{mn}} = 0, \quad \frac{\partial(U+V)}{\partial b_{mn}} = 0, \tag{11}$$

and

$$\frac{\partial(U+V)}{\partial c_{mn}} = 0.$$

Equations (11) result in a set of simultaneous, linear, algebraic equations that can be solved for the a_{mn}, b_{mn}, and c_{mn} which may then be substituted into Eqs. (10) to give the displacements for any x and θ.

The entire computational procedure that has been outlined above has been programmed for machine computation on the IBM 7094 for arbitrary shell dimensions and for load and thermal

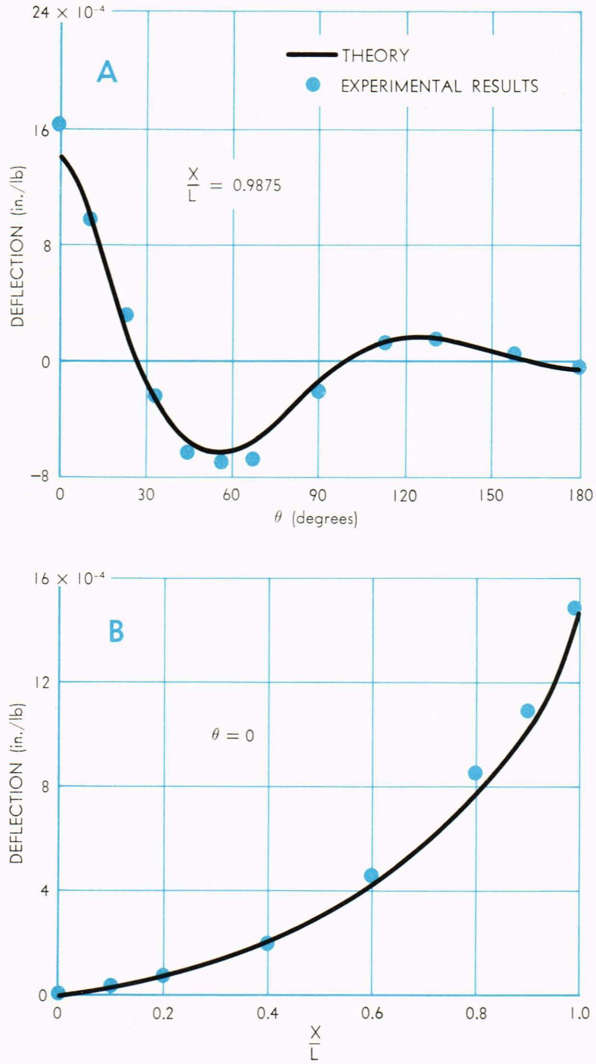

Fig. 5.—Comparison of theoretical and experimental results for a unit load applied at $x/L = 0.9875$, with the deflection (A) at $x/L = 0.9875$ versus the circumferential coordinate, and (B) at $\theta = 0$ versus the longitudinal coordinate.

distributions. Details of the program, which also computes stresses and angular displacements, may be found in Ref. 1. Machine running time for N in Eqs. (10), with $N = 15$, is ≈ 0.6 min.

Description of Tests

In order to obtain a comparison between the theory and the actual physical problem, tests were performed at uniform temperatures on a cylinder that was loaded by a single concentrated radial force. It is felt that the concentrated load provides a severe test of the theory. The dimensions of the test cylinder, having a thickened region at the free end, are shown in Fig. 2. The method of clamping is shown in Fig. 3, and an overall view of the test apparatus is given in Fig. 4. A radial load was applied successively at locations with x/L equal to 0.1, 0.2, 0.6 and 0.9875. (See Ref. 1).

Results

Typical theoretical and experimental results are shown in Figs. 5 and 6; note the difference in the vertical scales in these two figures. We see from these figures that the theory gives deformation modes that are similar in shape to those of the experimental data. However, in all cases the theory underestimates the maximum deflection. This would be expected from the Rayleigh-Ritz method because of the truncation of the series (Eqs. 10) that are used to approximate the solution. We believe that some of the difficulty also lies in the shell theory that was used since it does not include transverse shear deformations that are probably significant in the concentrated loading region.

From Fig. 5 it is observed that the agreement between theory and experiment is very good when the load is applied in the vicinity of the free end. However, agreement becomes progressively poorer as the load is moved toward the clamped end, as seen in Fig. 6. The deflection variation in the longitudinal direction is smooth for the load near the

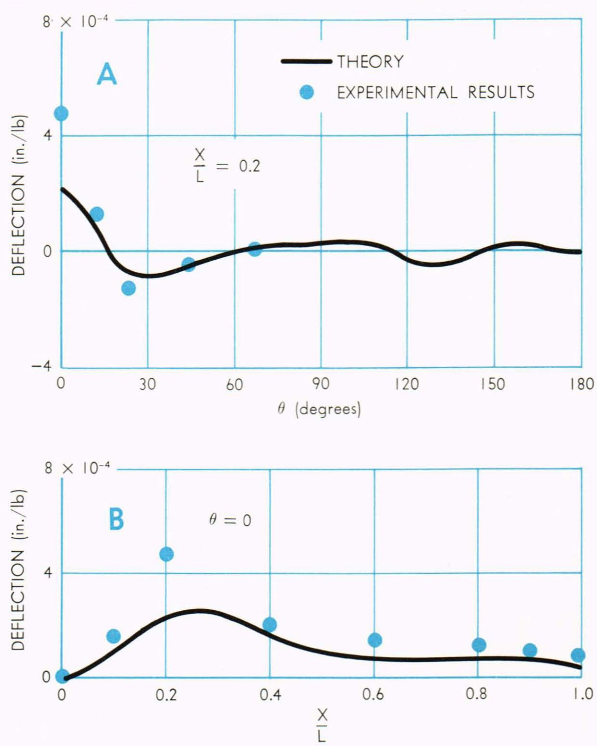

Fig. 6.—Comparison of theoretical and experimental results for a unit load applied at $x/L = 0.2$, with the deflection (A) at $x/L = 0.2$ versus the circumferential coordinate, and (B) at $\theta = 0$ versus the longitudinal coordinate.

free end but becomes more abrupt and localized for the load close to the clamped end. It would be expected, therefore, that for equivalent accuracies, N in Eqs. (10) would have to have a larger value for a load near the clamped end than would be required if the load were near the free end. It was found, however, that the conditioning of the simultaneous equations that must be solved for a_{mn}, b_{mn}, and c_{mn} was poorer for the loads near the clamped end and that this produced erratic results if N was made too large. That is, rounding off errors in the numerical solution of the simultaneous algebraic equations produced serious discrepancies in the results. Improved numerical methods for solving simultaneous equations would alleviate this difficulty. The values of N for the theoretical results of Figs. 5 and 6 are 15 and 12, respectively.

The last four terms inside the second integral of Eq. (2) are usually neglected in thin-shell theory. Computations with the similar expression for the strain energy produced results that differed only slightly from those using all of the terms in Eq. (2), with the latter case producing generally better agreement with the experimental results. It is expected that for a smoother loading, the difference in the two energy expressions would have a negligible effect. It is also felt that for a smoothly varying load, the agreement between theory and experiment, with $N = 15$, would be satisfactory for engineering computations and would be expected to produce better agreement than has been obtained for the concentrated load. This is evidenced by the satisfactory agreement of the displacements for the load near the free end where the longitudinal variation is smooth.

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