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Fuzzy age-dependent replacement policy and SPSA algorithm based-on fuzzy simulation

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Abstract

An increase in the performance of deteriorating systems can be achieved through the adoption of suitable maintenance policies. One of the most popular maintenance policies is the age-dependent replacement policy. In this paper, a fuzzy age-dependent replacement policy is considered in which the lifetimes of components are treated as fuzzy variables. To minimize the long-term expected cost per unit time, a programming model is formulated. Also, a theorem for optimal solution existence is proposed. In order to solve the proposed model, a fuzzy simulation technique is designed which estimates the expected value of the objective function. The simultaneous perturbation stochastic approximation (SPSA) algorithm is then used to determine a solution. Finally, a numerical example is presented to illustrate the effectiveness of this technique.

Keywords: Fuzzy variable; Fuzzy simulation; Maintenance policy; Replacement policy; SPSA algorithm

1. Introduction

Failure of a system during operation may have costly or safety-related consequences. However, if a suitable maintenance policy were to be adopted, the lifetime of the system could be extended considerably. Since the 1960s, there has been much focus on reliability theory in an attempt to reduce failures and extend system lifetimes. Central to this theory is the uncertainty of randomness, however, the environment in which real maintenance problems occur is often imprecise and the parameters which influence the decisions in the maintenance schedule cannot be assessed exactly. In addition, when the data is sparse, the use of statistical estimation to model a system may lead to inefficiency. As a result of this, fuzzy variables are more suitable for the characterization of system lifetimes.

Considerable work in the area of maintenance policies has been carried out. Zadeh [15–17] introduced the concepts of fuzzy sets and possibility measures and outlined the generalized theory of uncertainty in a much broader perspective. Bag and Samanta [2] defined strongly fuzzy convergent sequences, and provided fixed

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point theorems for fuzzy non-expansive mappings. Al-Najjar and Alsyouf [1] assessed the most popular maintenance approaches using a methodology based on fuzzy multiple-criteria decision-making evaluation. Chang [3] presented a fuzzy methodology for the replacement of equipment and discussed a fuzzy model with degradation parameters for determining fuzzy strategic replacement, and economic lives. Suresh [14] presented a fuzzy set model for maintenance policies of multistate equipment. Huang et al. [4] proposed a new approach using fuzzy dynamic programming for generator maintenance scheduling using multiple objectives and soft constraints expressed by fuzzy sets. Huang [5] also presented a genetically-evolved fuzzy system for maintenance scheduling of generators. The fuzzy system is formulated with respect to multiple objectives and soft constraints. Genetic algorithms are then applied to tune membership functions in the solution process. Huang et al. [6] discussed the problem of capital budgeting in a fuzzy environment and proposed two types of model using credibility to measure confidence level.

In this paper, we develop a novel method for the investigation of age-dependent replacement policies with fuzzy lifetimes. Section 2 reviews some of the properties of fuzzy variables and in Section 3, the age-dependent replacement policy is considered, and a model that uses fuzzy lifetimes is proposed. A fuzzy simulation technique is explored in Section 4, as it is almost impossible to define an analytical method which can compute the expected values of fuzzy variables. The SPSA algorithm based on fuzzy simulation is also proposed in order to discover an optimal solution. Finally, a numerical example is given to illustrate the proposed method.

2. Fuzzy variables

Let Θ be a nonempty set, and $P(\Theta)$ the power set of Θ. In order to present the axiomatic definition of possibility, Nahmias [10] and Liu [7] gave the following four axioms.

Axiom 1 $\text{Pos}\{\Theta\} = 1$.
Axiom 2 $\text{Pos}\{\phi\} = 0$.
Axiom 3 $\text{Pos}\{\bigcup_i \mathscr{A}_i\} = \sup_i \text{Pos}\{\mathscr{A}_i\}$ for any collection $\mathscr{A}_i$ in $P(\Theta)$.
Axiom 4 Let Θ_i be nonempty sets on which $\text{Pos}_i\{\cdot\}$ satisfies the first three axioms, $i = 1, 2, \ldots, n$, respectively, and $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$. Then

$$\text{Pos}\{\mathscr{A}\} = \sup_{(\theta_1,\theta_2,\ldots,\theta_n)\in\mathscr{A}} \text{Pos}_1\{\theta_1\} \wedge \text{Pos}_2\{\theta_2\} \wedge \cdots \wedge \text{Pos}_n\{\theta_n\}, \tag{1}$$

for each $\mathscr{A} \in P(\Theta)$.

Definition 1 (*Liu and Liu* [9]). Let Θ be a nonempty set, and $P(\Theta)$ the power set of Θ. Then the set function Pos is called a possibility measure, if it satisfies the first three axioms, and $(\Theta, P(\Theta), \text{Pos})$ is called a possibility space.

Definition 2. Let $(\Theta, P(\Theta), \text{Pos})$ be a possibility space, and $\mathscr{A}$ a set in $P(\Theta)$. Then the necessity measure of $\mathscr{A}$ is defined by:

$$\text{Nec}\{\mathscr{A}\} = 1 - \text{Pos}\{\mathscr{A}^c\}. \tag{2}$$

Definition 3 (*Liu and Liu* [9]). Let $(\Theta, P(\Theta), \text{Pos})$ be a possibility space, and $\mathscr{A}$ a set in $P(\Theta)$. Then the credibility measure of $\mathscr{A}$ is defined by:

$$\text{Cr}\{\mathscr{A}\} = \frac{1}{2}(\text{Pos}\{\mathscr{A}\} + \text{Nec}\{\mathscr{A}\}). \tag{3}$$

Definition 4. A fuzzy variable ξ is defined as a function from the possibility space $(\Theta, P(\Theta), \text{Pos})$ to the set of real numbers, and its membership function is derived by:

$$\mu_\xi(r) = \text{Pos}\{\theta \in \Theta | \xi(\theta) = r\}.$$

Definition 5 (*Liu and Liu* [9]). Let ξ be a fuzzy variable on the possibility space $(\Theta, P(\Theta), \mathrm{Pos})$. The expected value $E[\xi]$ is defined by:

$$E[\xi] = \int_0^{\infty} \mathrm{Cr}\{\xi \geqslant r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{\xi \leqslant r\}\mathrm{d}r, \tag{4}$$

provided that at least one of the two integrals is finite. In particular, if the fuzzy variable ξ is positive (i.e., $\mathrm{Pos}\{\xi \leqslant 0\} = 0$), then

$$E[\xi] = \int_0^{\infty} \mathrm{Cr}\{\xi \geqslant r\}\mathrm{d}r.$$

Remark 1 (*Liu and Liu* [9]). Let ξ be a discrete fuzzy variable with membership function $\mu(a_i) = \mu_i$ for $i = 1, 2, \ldots, n$, we assume that $a_1 < a_2 < \cdots < a_n$. Definition 5 implies that:

$$E[\xi] = \sum_{i=1}^{n} \omega_i a_i, \tag{5}$$

where the weights ω_i, $i = 1, 2, \ldots, n$ are given by:

$$\begin{aligned}
\omega_1 &= \frac{1}{2}\left(\mu_1 + \max_{1\leqslant j\leqslant n} \mu_j - \max_{1<j\leqslant n} \mu_j\right), \\
\omega_i &= \frac{1}{2}\left(\max_{1\leqslant j\leqslant i} \mu_j - \max_{1\leqslant j<i} \mu_j + \max_{i\leqslant j\leqslant n} \mu_j - \max_{i<j\leqslant n} \mu_j\right), \quad 2 \leqslant i \leqslant n-1, \\
\omega_n &= \frac{1}{2}\left(\max_{1\leqslant j\leqslant n} \mu_j - \max_{1\leqslant j<n} \mu_j + \mu_n\right).
\end{aligned}$$

Definition 6 (*Liu and Liu* [7]). The fuzzy variables $\xi_1, \xi_2, \ldots, \xi_n$ defined on the possibility space $(\Theta, P(\Theta), \mathrm{Pos})$ are said to be independent if and only if

$$\mathrm{Pos}\{\xi_i \in \mathscr{A}_i \quad i = 1, 2, \ldots, n\} = \min_{1\leqslant i\leqslant n} \mathrm{Pos}\{\xi_i \in \mathscr{A}_i\}$$

for any sets $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n$ in Θ.

Proposition 1 (Liu and Liu [8]). *Let ξ_1 and ξ_2 be two independent fuzzy variables with finite expected values. Then for any real numbers a and b, we have $E[a\xi_1 + b\xi_2] = aE[\xi_1] + bE[\xi_2]$.*

3. Fuzzy age-dependent replacement policy

One of the most popular maintenance policies is the age-dependent replacement policy. Using this policy, a unit is always replaced either at age T (where T is a constant) or following failure, whichever occurs first.

Let the positive fuzzy variable ξ_k be the lifetime of the kth renewed component, $k = 1, 2, \ldots$, respectively. Further, we assume that the system is as new following a replacement such that the fuzzy variables ξ_k have an identical membership function for $k = 1, 2, \ldots$ Thus, the length of the kth cycle is:

$$L_k(T) = \xi_k I_{(\xi_k \leqslant T)} + T I_{(\xi_k > T)}, \tag{6}$$

where $I_{(\cdot)}$ is the characteristic function of the event $(\cdot)$, and defined as:

$$I_{(\cdot)} = \begin{cases} 1, & \text{if event } (\cdot) \text{ has occurred,} \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Moreover, the cost of the kth cycle is:

$$C_k(T) = c_f I_{(\xi_k \leqslant T)} + c_p I_{(\xi_k > T)}, \tag{8}$$

where c_f and c_p are the respective costs of replacement upon failure and age T. Usually, we assume that $c_f \gg c_p$. If $c_f \leqslant c_p$, it is clear that we do not need to carry out any maintenance, rather another replacement when the system fails.

Define $S_0 = 0$ and

$$S_k = L_1(T) + L_2(T) + \cdots + L_k(T), \quad \forall k \geqslant 1. \tag{9}$$

It follows that S_k is the time of the kth replacement. Since $\{\xi_k, k \geqslant 1\}$ is a sequence of independent fuzzy variables with the same membership function, then, for any $t > 0$, it follows from Zhao and Liu [18] that

$$\mathrm{Pos}\{S_k \leqslant t\} = \mathrm{Pos}\left\{\xi_1 \leqslant \frac{t}{k}\right\}. \tag{10}$$

Eq. (10) still holds if "$\leqslant$" is replaced with "$\geqslant$", "$<$" or "$>$".

Let $N(t)$ be the number of replacements in time interval $(0, t]$. Then

$$N(t) = \max_{k \geqslant 0} \{k | 0 < S_k \leqslant t\}. \tag{11}$$

It is clear that $N(t)$ is a fuzzy variable, and its membership function is characterized by:

$$\mu_{N(t)}(k) = \mathrm{Pos}\{S_k \leqslant t < S_{k+1}\}.$$

Further, it also follows from (10) that:

$$\mu_{N(t)}(k) = \mathrm{Pos}\left\{\frac{t}{k+1} < L_1(T) \leqslant \frac{t}{k}\right\}. \tag{12}$$

We call $N(t)$ the *fuzzy renewal variable*.

Let $C(t)$ denote the total cost in $(0, t]$, i.e.,

$$C(t) = \sum_{k=1}^{N(t)} C_k(T). \tag{13}$$

Definition 7. The long-term expected cost per unit time under the age-dependent replacement policy is defined as:

$$C_a(T) = \lim_{t \to \infty} \frac{\text{the expected cost in } (0, t]}{t}. \tag{14}$$

Remark 2. It is easy to see that

$$C_a(T) = \lim_{t \to \infty} \frac{E[C(t)]}{t}. \tag{15}$$

Also, in [18], if $\{L_k(T), k \geqslant 1\}$ is a sequence of positive fuzzy interarrival times with the same membership function, $C_k(T)$ is the positive fuzzy reward with the same membership function associated with the kth interarrival time $L_k(T)$, then

$$C_a(T) = E\left[\frac{C_1(T)}{L_1(T)}\right] = E\left[\frac{c_f}{\xi_1} I_{(\xi_1 \leqslant T)} + \frac{c_p}{T} I_{(\xi_1 > T)}\right], \tag{16}$$

provided that $E[\frac{C_1(T)}{L_1(T)}]$ is finite.

The aim is to find an optimal value T such that $C_a(T)$ is minimized, and also one which can be described by the following model:

$$\begin{cases} \min & E\left[\frac{C_1(T)}{L_1(T)}\right], \\ \text{s.t.} & T > 0, \end{cases} \tag{17}$$

where $L_1(T) = \xi_1 I_{(\xi_1 \leqslant T)} + TI_{(\xi_1 > T)}$ is the length of the first cycle, $C_1(T) = c_f I_{(\xi_1 \leqslant T)} + c_p I_{(\xi_1 > T)}$ the cost of the first cycle, and ξ_1 the lifetime of the first renewed component.

Theorem 1. *Let $C_a(T)$ be the long-term expected cost per unit time defined by Definition 7. $C_a(T)$ is a convex function with respect to T, if $\frac{\mathrm{d}C_a(T)}{\mathrm{d}T}, \frac{\mathrm{d}^2 C_a(T)}{\mathrm{d}T^2}$ exist, and $\frac{\mathrm{d}^2\mathrm{Cr}\{\xi_1 \leqslant T\}}{\mathrm{d}T^2} > 0$.*

Proof 1. It follows from (16) that

$$
\begin{aligned}
C_a(T) &= E\left[\frac{c_f}{\xi_1} I_{(\xi_1 \leqslant T)} + \frac{c_p}{T} I_{(\xi_1 > T)}\right] \\
&= \int_0^{+\infty} \mathrm{Cr}\left\{\frac{c_f}{\xi_1} \cdot I_{(\xi_1 \leqslant T)} + \frac{c_p}{T} \cdot I_{(\xi_1 > T)} > r\right\} \mathrm{d}r \\
&= \int_0^{+\infty} \mathrm{Cr}\left\{\frac{c_f}{\xi_1} \cdot I_{(\xi_1 \leqslant T)} > r\right\} \mathrm{d}r + \int_0^{+\infty} \mathrm{Cr}\left\{\frac{c_p}{T} \cdot I_{(\xi_1 > T)} > r\right\} \mathrm{d}r \\
&= \int_0^{+\infty} \mathrm{Cr}\left\{\left\{\frac{c_f}{\xi_1} > r\right\} \bigcap \{\xi_1 \leqslant T\}\right\} \mathrm{d}r + \frac{c_p}{T} \cdot \mathrm{Cr}\{\xi_1 > T\} \\
&= \int_{\frac{c_f}{T}}^{+\infty} \mathrm{Cr}\left\{\xi_1 < \frac{c_f}{r}\right\} \mathrm{d}r + \int_0^{\frac{c_f}{T}} \mathrm{Cr}\{\xi_1 \leqslant T\} \mathrm{d}r + \frac{c_p}{T} \cdot \mathrm{Cr}\{\xi_1 > T\} \\
&= c_f \int_0^T \frac{\mathrm{Cr}\{\xi_1 < t\}}{t^2} \mathrm{d}t + \frac{c_f}{T} \mathrm{Cr}\{\xi_1 \leqslant T\} + \frac{c_p}{T} \cdot \mathrm{Cr}\{\xi_1 > T\} \\
&= c_f \int_0^T \frac{\mathrm{Cr}\{\xi_1 < t\}}{t^2} \mathrm{d}t + (c_f - c_p) \frac{\mathrm{Cr}\{\xi_1 \leqslant T\}}{T} + \frac{c_p}{T}.
\end{aligned} \tag{18}
$$

Following (18), we can obtain the first derivative of $C_a(T)$ with respect to T:

$$
\begin{aligned}
\frac{\mathrm{d}C_a(T)}{\mathrm{d}T} &= c_f \frac{\mathrm{Cr}\{\xi_1 < T\}}{T^2} + (c_f - c_p) \frac{\frac{\mathrm{dCr}\{\xi_1 \leqslant T\}}{\mathrm{d}T} T - \mathrm{Cr}\{\xi_1 \leqslant T\}}{T^2} - \frac{c_p}{T^2} \\
&= c_p \frac{\mathrm{Cr}\{\xi_1 < T\}}{T^2} + (c_f - c_p) \frac{\mathrm{dCr}\{\xi_1 \leqslant T\}}{T\mathrm{d}T} - \frac{c_p}{T^2} \\
&= \frac{1}{T^2}\left[(c_f - c_p) T \frac{\mathrm{dCr}\{\xi_1 \leqslant T\}}{\mathrm{d}T} - c_p \mathrm{Cr}\{\xi_1 > T\}\right]
\end{aligned} \tag{19}
$$

and the second derivative of $C_a(T)$ with respect to T:

$$
\begin{aligned}
\frac{\mathrm{d}^2 C_a(T)}{\mathrm{d}T^2} &= c_f \frac{\frac{\mathrm{dCr}\{\xi_1 \leqslant T\}}{\mathrm{d}T} T^2 - 2\mathrm{Cr}\{\xi_1 < T\} T}{T^4} + (c_f - c_p) \frac{\frac{\mathrm{d}^2\mathrm{Cr}\{\xi_1 \leqslant T\}}{\mathrm{d}T^2} T^3 - 2 \frac{\mathrm{dCr}\{\xi_1 \leqslant T\}}{\mathrm{d}T} T + 2\ \mathrm{Cr}\{\xi_1 \leqslant T\} T}{T^4} + 2\frac{c_p}{T^3} \\
&= \frac{1}{T^3}\left[(c_f - c_p) \frac{\mathrm{d}^2\mathrm{Cr}\{\xi_1 \leqslant T\}}{\mathrm{d}T^2} T^2 + (c_f - 2c_p) \frac{\mathrm{dCr}\{\xi_1 \leqslant T\}}{\mathrm{d}T} T + 2c_p(1 - \mathrm{Cr}\{\xi_1 < T\})\right]
\end{aligned} \tag{20}
$$

Note that $\frac{\mathrm{dCr}\{\xi_1 \leqslant T\}}{\mathrm{d}T} > 0$, and $(c_f - 2c_p) > 0$. Thus, we can obtain $\frac{\mathrm{d}^2 C_a(T)}{\mathrm{d}T^2} > 0$, which implies that $C_a(T)$ is a convex function. Thus, the proof is complete. □

Corollary 1. *Let $C_a(T)$ be the long-term expected cost per unit time defined by Definition 7. There is one optimal solution T^* which minimizes $C_a(T)$, if ξ_1 is an L–R fuzzy variable.*

Proof 2. Let ξ_1 be an L–R fuzzy variable, usually written as:

$$
\mu_\xi(x) = \begin{cases} L[(m - x)/\alpha] & x \leqslant m, \alpha > 0 \\ R[(x - m)/\beta] & x > m, \beta > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{21}
$$

where L and R are decreasing and continuous functions from $[0,1]$ to $[0,1]$ satisfying $L(0)=R(0)=1$ and $L(1)=R(1)=0$. □

There are two cases:
Case 1. $T>m$. It follows (19) that:

$$\begin{aligned}\frac{\mathrm{d}C_a(T)}{\mathrm{d}T} &= \frac{1}{T^2}\left[(c_f-c_p)T\frac{\mathrm{dCr}\{\xi_1\leqslant T\}}{\mathrm{d}T}-c_p\mathrm{Cr}\{\xi_1>T\}\right]\\ &\geqslant \frac{1}{T^2}\left[(c_f-c_p)m\ Cr\{\xi_1=T\}-c_pm\mathrm{Cr}\{\xi_1=T\}\right]\geqslant\frac{1}{T^2}(c_f-2c_p)m\ Cr\{\xi_1=T\}\geqslant 0.\end{aligned} \tag{22}$$

It can now be seen that $C_a(T)$ is a monotonic increasing function, and $C_a(m)\leqslant C_a(T)$ for all $T>m$.
Case 2. $T\leqslant m$. It is clear that $\frac{\mathrm{d}^2C_a(T)}{\mathrm{d}T^2}>0$ holds, and it follows from Theorem 1 that $C_a(T)$ is a convex function and has one optimal solution T^* which minimizes $C_a(T)$.

Minimizing $C_a(T)$ with respect to the above cases, there is one optimal solution T^* which minimizes $C_a(T)$, if ξ_1 is an L–R fuzzy variable. Thus, the proof is complete.

Remark 3. Theorem 1 and Corollary 1 give two sufficient conditions for the existence of solutions of Model (17). However, this theorem can not provide a method of finding such a solution. In the following section, a method will be proposed to solve this.

4. SPSA algorithm based on fuzzy simulation and numerical experiment

It is difficult to compute the membership function of $\frac{C_1(T)}{L_1(T)}$ using an analytical method due to its complexity. This is because it is practically impossible to design an analytical method to compute $E[\frac{C_1(T)}{L_1(T)}]$. Liu and Liu [9] proposed a method to estimate the expected value of a fuzzy variable, and we also employ this method to estimate that value.

For any given time T, the fuzzy simulation for estimating $E[\frac{C_1(T)}{L_1(T)}]$ is given as follows:

1. Calculate $\mu_k=\mu_{\frac{C_1(T)}{L_1(T)}}(x_k)$ for $k=1,2,\ldots,n$. It is difficult to compute μ_k directly, and the following method is proposed to solve this problem. Firstly, generate a sequence of $\{\theta_k\}$ from Θ such that $\mathrm{Pos}\{\theta_k\}>0$, $k=1,2,\ldots,n$, respectively. Thus, n real numbers $\xi_1(\theta_k)$ can be obtained with the membership degree $\mu_k=\mu_{\xi_1}(\xi_1(\theta_k))$. Let
$$x_k=\frac{c_pI_{(\xi_1(\theta_k)\leqslant T)}+c_fI_{(\xi_1(\theta_k)>T)}}{\xi_1(\theta_k)I_{(\xi_1(\theta_k)\leqslant T)}+TI_{(\xi_1(\theta_k)>T)}}.$$
If x_i and x_j have the same values, remove x_j from the result sequence and set $\mu_i=\max(\mu_i,\mu_j)$.
2. Employ $\sum_{k=1}^{n}x_kw_k$ to estimate $E[\frac{C_1(T)}{L_1(T)}]$, where
$$w_1=\frac{1}{2}\left(\mu_1+\max_{1\leqslant j}\mu_j-\max_{1<j}\mu_j\right),$$
$$w_k=\frac{1}{2}\left(\max_{1\leqslant j\leqslant k}\mu_j-\max_{1\leqslant j<k}\mu_j+\max_{k\leqslant j}\mu_j-\max_{k<j}\mu_j\right),\quad 2\leqslant k\leqslant n.$$
In this step, $x_1<x_2<\cdots<x_n$ is needed, sorted in ascending order, and the sequence of μ_k should be modified respectively.

Theorem 1 and Corollary 1 give two sufficient conditions for the existence of solutions to Model (17). This type of problem can be resolved through the use of gradient-based algorithms. However, the relationship between the objective function and the decision variables is not often detailed enough to ensure that the gradient of $C_a(T)$ with respect to T always exists. Here, a kind of recursive optimization algorithm based on an approximation to the gradient is used to solve this.

The SPSA algorithm ([11,12]) is based on an easily implemented and highly efficient gradient approximation that relies on measurements of the objective function. However, it does not rely on measurements of the gradient of the objective function [13]. It can therefore be used to solve optimization problems where it is difficult or impossible to obtain the gradient of the objective function with respect to the parameters being optimized.

SPSA algorithm based on fuzzy simulation

Step 1. *Initiation and parameter selection* Set the counter index $k = 0$, and randomly choose an initial value $\widehat{T}_0$ of T (where T is the decision variable). Then, select nonnegative parameters a, c, A, α, and γ in the SPSA gain sequence $a_k = a/(A + k + 1)^\alpha$ and $c_k = c/(k + 1)^\gamma$. Usually α and γ are taken to be 0.602 and 0.101, respectively. The parameters a, A and c can be determined by specific situations ([11,12]).

Step 2. *Generation of simultaneous perturbation variables* Generate a sequence of random perturbation variables $\{\Delta_k\}$, which are independently generated from a zero-mean probability distribution satisfying the conditions in [11]. Here, we randomly generate Δ_k in $[-1, 1]$, and the distribution function of Δ_k is defined as follows:

$$F(\Delta_k) = \begin{cases} 0, & \Delta_k \leqslant -1 \\ \frac{\Delta_k+1}{2}, & -1 < \Delta_k \leqslant 1 \\ 1, & \Delta_k > 1. \end{cases}$$

Step 3. *Loss function evaluation* Calculate the two measurements of objective functions $C_a(\widehat{T}_k + c_k\Delta_k)$ and $C_a(\widehat{T}_k - c_k\Delta_k)$ by fuzzy simulation as shown above.

Step 4. *Gradient approximation* Calculate the simultaneous perturbation approximation to the unknown gradient:

$$\hat{g}_k(\widehat{T}_k) = \frac{C_a\left(\widehat{T}_k + c_k\Delta_k\right) - C_a\left(\widehat{T}_k - c_k\Delta_k\right)}{2c_k\Delta_k}.$$

Step 5. *Update estimate* Set $\widehat{T}_{k+1} = \widehat{T}_k - a_k\hat{g}_k\left(\widehat{T}_k\right)$.

Step 6. *Iteration or termination* Return to Step 2 with $k + 1$ replacing k. Terminate the algorithm and return $\widehat{T}_k$, if there is little change in several successive iterations, or the maximum allowable number of iterations has been reached. Thus, $\widehat{T}_k$ is reported as the optimal value of T.

Example 1. Let $\xi_1 = (0, 4, 7, 15)$ be the lifetime of the first component, $c_f = 1000$ and $c_p = 300$. Then the long-term expected cost per unit time is:

$$C_a(T) = E\left[\frac{C_1(T)}{L_1(T)}\right],$$

where, $L_1(T) = \xi_1 I_{(\xi_1 \leqslant T)} + TI_{(\xi_1 > T)}$ and $C_1(T) = 1000I_{(\xi_1 \leqslant T)} + 300I_{(\xi_1 > T)}$.

The purpose is to find an optimal T such that $C_a(T)$ is minimized, as defined by the following mathematical model:

$$\begin{cases} \min & E\left[\dfrac{1000I_{(\xi_1 \leqslant T)} + 300I_{(\xi_1 > T)}}{\xi_1 I_{(\xi_1 \leqslant T)} + TI_{(\xi_1 > T)}}\right] \\ \text{s.t.} & T > 0. \end{cases} \tag{23}$$

In this example, it is clear that ξ_1 is an L–R fuzzy variable and the objective function in Model (23) is a convex function with respect to the decision variable T and has only one optimal solution.

According to [11,12], the coefficients are set such that: $A = 100$, $c = 0.2$, $\alpha = 0.602$, $\gamma = 0.101$, and 20 initial values of T_0 are randomly generated from each of the four parts. The SPSA based on fuzzy simulation is run for 1000 cycle iterations with the generated initial values, for the results shown in Table 1. Compared with the

Table 1
Results for SPSA based on fuzzy simulation

T_0	T^*	$C_a(T^*)$	T_0	T^*	$C_a(T^*)$
0.1286	6.5011	79.2698	0.3001	6.7355	76.8245
0.5228	7.1481	78.2685	1.1156	6.5743	76.2603
1.3720	6.7702	76.9458	2.6794	6.8600	77.2601
3.5336	76.8935	78.5687	3.8293	7.4852	79.4485
4.0536	6.8189	77.1162	4.1000	6.9265	76.0041
4.7838	6.6786	76.6252	5.4002	7.3575	79.0015
6.7202	6.5348	76.1220	7.2883	6.5085	76.0300
9.4014	7.1267	78.1937	9.7223	6.9858	77.7006
12.0327	6.7729	76.9555	12.8636	7.3021	78.8076
14.0135	6.5914	76.3201	14.7794	6.9480	77.5680

solutions obtained, $T^* = 6.9265$, $C_a(T^*) = 76.0041$ with the initial value of $T_0 = 4.1000$ is reported as the optimal solution. Following Table 1, we can compute the mean value $\overline{T^*} = 6.8945$, $\overline{C_a(T^*)} = 77.3808$, and the variance $\mathrm{Var}(T^*) = 0.096$, $\mathrm{Var}(C_a(T^*)) = 1.1675$. The results imply that SPSA based on fuzzy simulation is a suitable method for solving such problems.

The convergence procedure with an initial value of $\widehat{T}_0 = 4.1000$ is presented in Fig. 1. Fig. 2 shows the convergence process of the solutions for various iteration times N with the initial value of $\widehat{T}_0 = 4.1000$. From this figure, it can be seen that the curve tends to plateau after 70 cycles using SPSA based on fuzzy simulation. Fig. 3 shows the graph of optimal solutions T^* for different initial values.

Example 2. Let ξ_1 be the lifetime of the first component defined as $\xi_1 = \eta^2 + 9$ with $\eta = (0, 3, 6)$, and $c_f = 200$, $c_p = 30$. Then the long-term expected cost per unit time is $C_a(T) = E[\frac{C_1(T)}{L_1(T)}]$, where $L_1(T) = \xi_1 I_{(\xi_1 \leqslant T)} + T I_{(\xi_1 > T)}$ and $C_1(T) = 200 I_{(\xi_1 \leqslant T)} + 30 I_{(\xi_1 > T)}$.

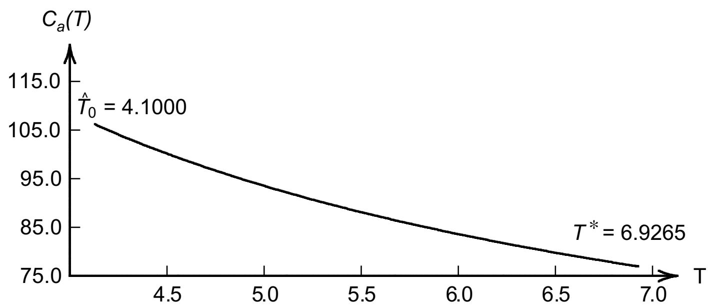

Fig. 1. The convergence procedure with the initial value $\widehat{T}_0 = 4.1000$.

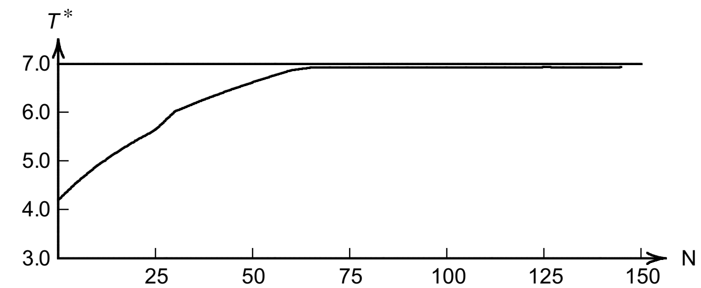

Fig. 2. The solutions of different iterations with the initial value $\widehat{T}_0 = 4.1000$.

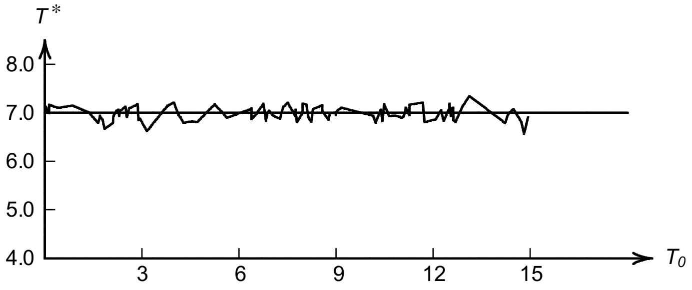

Fig. 3. The solutions with different initial values.

In order to find the optimal value T, the following mathematical model is established:

$$\begin{cases} \min & E\left[\frac{200_{(\xi_1 \leqslant T)}+30I_{(\xi_1>T)}}{\xi_1 I_{(\xi_1 \leqslant T)}+TI_{(\xi_1>T)}}\right] \\ \text{s.t.} & T>0. \end{cases}$$

Then, it can be obtained that:

$$\mathrm{Cr}\{\xi_1 < T\} = \left\{ \begin{array}{ll} 0, & 0 \leqslant T < 9 \\ \frac{\sqrt{T-9}}{6}, & 9 \leqslant T < 45 \\ 1, & 45 \leqslant T. \end{array} \right\} \tag{24}$$

Following (24), it is clear that ξ_1 is also a L–R fuzzy variable and that $C_a(T)$ is a convex function (Corollary 1), thus, there is at least one optimal solution. The parameters are set such that: $A = 100$, $c = 0.2$, $\alpha = 0.602$, $\gamma = 0.101$ and the proposed algorithm is run with stochastic samples for 20 iterations. The results obtained

Table 2
Results obtained using SPSA based on fuzzy simulation

T_0	T^*	$C_a(T^*)$	T_0	T^*	$C_a(T^*)$
9.0417	17.0023	6.0034	9.3087	17.4711	6.7067
10.2547	18.2963	7.9444	11.6774	17.1487	6.2231
12.2927	17.5405	6.8107	15.4305	17.7201	7.0800
17.4806	17.5106	6.7658	18.1903	18.9706	8.9558
18.7287	18.8685	6.9567	18.8276	18.1662	8.8027
20.4811	17.3572	6.5358	21.9610	18.7151	8.5727
25.1284	17.0697	6.1045	26.4919	17.0172	6.0257
31.5633	18.2535	7.8802	32.3335	17.9718	7.4576
37.8784	17.5460	6.8189	39.8725	18.6044	8.4065
42.6324	17.1829	6.2743	44.4704	17.8960	7.3440

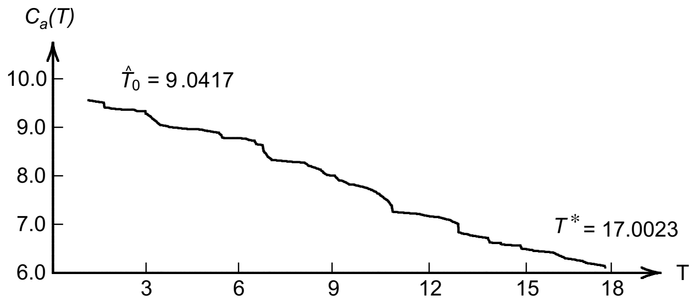

Fig. 4. The convergence procedure with the initial value $\widehat{T}_0 = 1.5$.

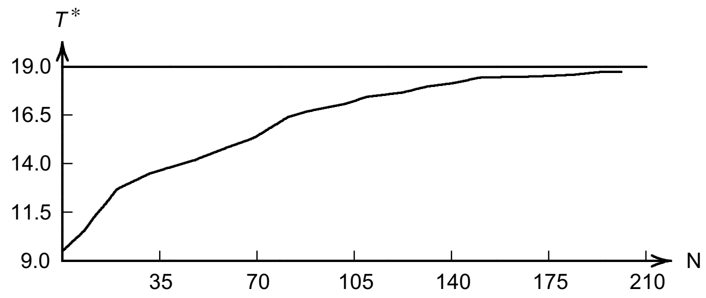

Fig. 5. The solutions with different iterations.

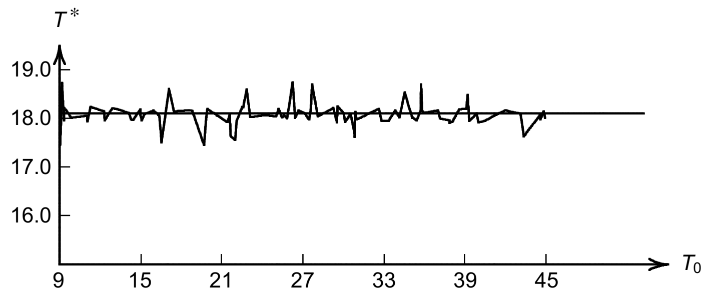

Fig. 6. The solutions with different initial values.

are shown in Table 2. And, it can be seen that the mean value $\overline{T^*} = 17.789$, $\overline{C_a(T^*)} = 7.18352$, the variance $\mathrm{Var}(T^*) = 0.381228$, $\mathrm{Var}(C_a(T^*)) = 0.857762$, and $C_a(T^*) = 6.0034$ are reported as the optimal solution with $T^* = 17.0023$.

Fig. 4 shows the convergence procedure with the initial value $\widehat{T}_0 = 9.0417$. The convergence procedure for different iterations when the initial value $T_0 = 9.0417$ can be seen in Fig. 5. From this figure it can also be seen that the curve tends to plateau after 200 cycle iterations of the SPSA based on fuzzy simulation. Fig. 6 graphs the optimal solutions T^* for different initial values.

5. Conclusions

In this paper, the age-dependent replacement policy with fuzzy lifetimes was discussed. The systems in which components' lifetimes are characterized by independent nonnegative fuzzy variables with identical membership functions and a model to describe this maintenance policy were presented. Numerical results demonstrate that the expected cost of renewal per unit time is equal to the expected value of cost in the first maintenance cycle (per unit time). In order to obtain the expected cost of maintenance and find the optimal solution of the maintenance cycle to minimize the long-term expected cost per unit time, the SPSA algorithm based on fuzzy simulation was presented to solve the proposed model.

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