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EXTENDED ADAPTIVE ROBBINS-MONRO PROCEDURE USING SIMULTANEOUS PERTURBATION FOR A LEAST-SQUARE APPROXIMATION PROBLEM.

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Abstract. Consider a static multiple-input multiple-output unknown system. Suppose that the dimension of the output of the system is greater than the one of the input of the system. Extended Adaptive Robbins-Monro procedure is appropriate for finding a least-square approximation input parameter, which minimizes the output error. However, if the system is large-dimension, a number of observations for the system increases. In this paper, we present an extended adaptive Robbins-Monro procedure using the simultaneous perturbation. In this algorithm, the number of measurements of the system does not increase no matter how the dimension of the system increases.

Keywords. Adaptive Robbins-Monro Procedure, Simultaneous Perturbation, Least-square Approximation, Stochastic Approximation

1. INTRODUCTION

We consider a static multiple-input multiple-output unknown system. Let $u \in \Re^n$ (n-dimensional Euclidean space) and $y \in \Re^m$ be an input and an output of the system, respectively. $f(u)$ represents a system characteristic;

$$y = f(u) \tag{1}$$

Our problem is to find an optimal input u_* that satisfies the following relation.

$$r = f(u_*) \tag{2}$$

Where, $r \in \Re^m$ is a desired output of the system.

We can measure the output y under noisy environment. That is, measurements are corrupted by a certain observation noise $\psi \in \Re^m$ with zero mean. Therefore, observed output z is as follows;

$$z = y + \psi \tag{3}$$

If the dimension of the output is equal to the one of the input, we can apply the usual Robbins-Monro stochastic approximation. Meanwhile, we would like to handle more general problem. That is, we considered the case that the dimension of the output is greater than the one of the input in this paper.

Then, let us introduce the following squared output error as an evaluation function $J(u)$.

$$\begin{aligned} J(u) &= \mathrm{E}\{(z-r)^T(z-r)\} \\ &= (y-r)^T(y-r) + \mathrm{E}(\psi^T\psi) \end{aligned} \tag{4}$$

In this paper, we find the least-square approximation input u_* that minimizes the above evaluation function, that is, which minimizes the squared error of the output.

The extended adaptive Robbins-Monro stochastic approximation( ) proposed by the authors is applicable to this kind of problems.

$$u_{t+1} = u_t - \frac{1}{t} H_t^{-1} K_t^T \left[\left\{ \frac{1}{2n} \sum_{i=1}^{n} (z_t^{+i} + z_t^{-i}) \right\} - r \right]$$

Where,

$$H_t = \begin{cases} K_t^T K_t & \text{if } \det(K_t^T K_t) \neq 0 \\ I & \text{if } \det(K_t^T K_t) = 0 \end{cases} \tag{5}$$

$$K_{t+1} = \frac{t}{t+1} K_t + \frac{1}{t+1} \frac{Z_t^+ - Z_t^-}{2c_t} \tag{6}$$

Where, I denotes an identity matrix. u_t is an estimator of u_* at the t-th iteration. We add perturbations $\pm c_t$ to the i-th element of u_t, and make observations at the points. The observation vectors $z_t^{\pm i}$ denote the observed values at these points. That is,

$$z_t^{\pm i} = f(u_t \pm e^i c_t) + \psi_t^{\pm i} \quad (i = 1, \cdots, n) \tag{7}$$

$\psi_t^{\pm i}$ denote the observation noise vectors. e^i denotes a fundamental vector whose i-th element is 1 and the other are all 0. Moreover, the observation matrices $Z_t^{\pm}$ of Eq.(6) are defined as follows, as same as the ordinary adaptive Robbins-Monro (ARM) procedure (Nevel'son et al.(1973a), Nevel'son et al.(1973b), Maeda et al.(1990a), Maeda et al.(1990b)).

$$Z_t^{\pm} = (z_t^{\pm 1}, \cdots, z_t^{\pm n})$$

This algorithm is an extension of the usual ARM procedure. The matrix K_t denotes an estimator of $f'(u_t)$. The gain matrix $H_t^{-1} K_t^T$ is used in order to convert m dimensional observation vector into n dimensional vector. Moreover, this matrix improves the asymptotic convergence rate of the algorithm. In this algorithm, we use the perturbations to estimate the derivative as with the usual ARM procedure. Therefore, such a procedure need additional observations.

On the other hand, a similar extension of the usual Robbins-Monro procedure was also proposed by one of the authors (Maeda et al.(1991)).

$$u_{t+1} = u_t - \frac{1}{t} K_t^{-1} \{ z_t - r \} \tag{8}$$

Where, gain matrix K_t is $m \times n$ matrix. Moreover,

$$z_t = f(u_t) + \psi_t \tag{9}$$

Instead of the estimator of $f'(u_t)$, this algorithm uses a constant $m \times n$ gain matrix K_t. However, how to determine the gain matrix is serious in this algorithm.

Generally, gain matrix is used to improve asymptotic convergence rate. On the other hand, since the dimensions of the input and the output are different in our problem, the gain matrix should have the another role. That is, in these algorithms, the gain matrices are utilized in order to not only improve asymptotic convergence rate of u_t but also convert m dimensional observed vector into n dimensional vector. We have to determine the suitable gain matrix so that the gain matrix converts the observed vector properly. The algorithm of Eqs.(5) and (6) is an adaptive Robbins-Monro type of resolutions. However, this algorithm needs additional observations. If the dimension n increases, this algorithm is not acceptable.

The second algorithm (8) defeats the problem of the increase of the observations. However, the condition of the gain matrix that ensures the convergence is closely related to the shape of $f(\cdot)$. Therefore, it is generally difficult to determine the gain matrix exactly, because $f(\cdot)$ is unknown.

The idea of "simultaneous perturbation" .contrived by J. C. Spall is useful to solve the above problem (Spall(1992)). J. C. Spall used the simultaneous perturbation to Kiefer-Wolfowitz-type of the stochastic approximation (Spall(1992)). He also compared the efficiency between an algorithm via the simultaneous perturbation and the usual Kiefer-Wolfowitz procedure. Consequently, he pointed out that the algorithm via the simultaneous perturbation is superior.

This idea ia also applicable to our extended ARM procedure of Eqs.(5) and (6). Using the simultaneous perturbation, we can keep the number of observations constant no matter how n (the dimension of the input) increases.

2. MAIN ALGORITHM AND CONVERGENCE THEOREM

We propose an extended ARM procedure using the simultaneous perturbation. First of all, we define observation vectors with perturbation vector ξ_t.

$$z_t^{\pm} = f(u_t \pm \xi_t c_t) + \psi_t^{\pm}$$

Then, an observation matrix Z_t is defined as follows:

$$Z_t = \left(\frac{z_t^{i+} - z_t^{i-}}{2 c_t \xi_t^{(j)}} \right)_{ij} \tag{10}$$

Where, $z_t^{i\pm}$ and $\xi_t^{(j)}$ denote the i-th element of the vector $z_t^{\pm}$ and the j-th element of the perturbation vector ξ_t, respectively.

Then, main algorithm is as follows:

$$u_{t+1} = u_t - \frac{1}{t} H_t^{-1} K_t^T \left[\left\{ \frac{1}{2}(z_t^+ + z_t^-) \right\} - r \right]$$

Where,

$$H_t = \begin{cases} K_t^T K_t & \text{if } \det(K_t^T K_t) \neq 0 \\ I & \text{if } \det(K_t^T K_t) = 0 \end{cases} \tag{11}$$

$$K_{t+1} = \frac{t}{t+1} K_t + \frac{1}{t+1} Z_t \tag{12}$$

We explain the procedure of our algorithm. First, we make random perturbation vector $\xi_t \in \Re^n$. Each component of this vector is mutually independent and with zero-mean. c_t is a positive deterministic sequence defined in the condition [6].

Second, we make two observations at two points $(u_t \pm \xi_t c_t)$. Observation noise $\psi_t^{\pm}$ should obey the condition [5]. Then, we have $z_t^{\pm}$. Using these values, we can construct the observation matrix (10). Instead of $2n$ times observations of Eq.(7), we can obtain the observation matrix Z_t by twice observations of $f(\cdot)$ of Eq.(10). This matrix Z_t is an estimator of $f'(\cdot)$ by a difference approximation in a sense. However, a length of the perturbation of each component for the difference approximation is different each other and determines randomly. Compared with the usual extended ARM procedure, only two observations are required to estimate $f'(\cdot)$. Even if the dimension of the input increases, there is no increase of the measurements.

Finally, Eq.(12) calculates an arithmetical mean of Z_t. Every 2 times observations, u_t and K_t are modified by Eqs.(11) and (12). The notation t represents this iteration. Therefore, total number of observations at the t-th iteration is $2t$ times.

We assume the following conditions.

Conditions

[1] $f(\cdot)$ is two times continuously differentiable. Moreover, the dimension n of the input u and the dimension m of the output y satisfy the following relation.

$$m \geq n$$

[2] For arbitrarily $u^a, u^b, u^c, u^d \in \Re^n$, there exist positive definite matrices M_1 and M_2 such that

$$f'(u^a)^T f'(u^b) > M_1,$$

$$\begin{aligned} & f'(u^a)^T f'(u^b) f'(u^c)^T f'(u^d) \\ + \ & f'(u^d)^T f'(u^c) f'(u^b)^T f'(u^a) \\ & > M_2 \end{aligned} \tag{13}$$

[3] For an arbitrarily input $u \in \Re^n$, there exists a positive number m_1 such that

$$\|f'(u)\| < m_1$$

Where, the matrix norm $\|\cdot\|$ denotes the Euclidean norm. [4] For an arbitrarily $u \in \Re^n$ and i, j, k,

$$\left| \frac{\partial^2 f^{(i)}(u)}{\partial u^{(j)} \partial u^{(k)}} \right| \leq m_2$$

Where, $f^{(i)}$ represents the i-th element of $f(u)$. $u^{(j)}$ and $u^{(k)}$ denote the j-th and k-th components of the vector u. [5] $\mathrm{E}(\psi_t^{\pm i}) = o$, $\mathrm{E}(\psi_t^{\pm i} \psi_t^{\pm iT}) = V$. Moreover, $\psi_t^{\pm i}$ are independent to each observation. [6] There exist positive numbers C_1, C_2 and ε such that

$$C_1 t^{-\epsilon} \leq c_t \leq C_2 t^{-\epsilon}, 1/4 < \varepsilon < 1/2.$$

[7] $\xi_t^{(i)}$ $(i = 1, \cdots, n)$ are i.i.d. and symmetrically distributed about zero. Moreover, there exist positive numbers x_0, x_1 and x_2 such that

$$\mathrm{E}(\xi_t^{(i)}) = 0, \ \mathrm{E}|(\xi_t^{(i)})^{-1}| \leq x_1,$$

$$\mathrm{E}(\xi_t^{(i)})^{-1} \leq x_2, \ |\xi_t^{(i)}| < x_0 \text{ with probability 1.}$$

The conditions [1]~[3] are related to the characteristic of the unknown system. The condition [2] restricts the shape of the function $f(\cdot)$. When the system is a single-input single-output system and the sign of $f'(u)$ does not change with respect to u, this condition hold. The condition [5] prescribes the properties of the observation noise. The condition [6] relates to a magnitude of the perturbation. c_t must tends to 0 faster than $t^{-1/4}$ and slower than $t^{-1/2}$. The conditions [1]~[6] except [4] are the same as the ones of the usual extended ARM procedure (Maeda et al.(1992)). On the other hand, the conditions [4] and [7] that relate to the simultaneous perturbation are the same conditions that the reference (Spall(1992)) uses. These conditions are required to cancel an error of an approximation by the simultaneous perturbation.

Under these conditions, we can obtain the following theorem.

Theorem 2.1 If the condition [1]~[7] hold. Then, (11) and (12) have the following properties with probability 1.

$$u_t \to u_*,$$

$$K_t \to f'(u_*),$$

$$H_t \to f'(u_*)^T f'(u_*) \qquad (t \to \infty).$$

The first assertion is the main result that ensures the convergence of u_t to u_*. The second and the third assertions of K_t and H_t are required to improve the convergence rate of u_t and to determine the proper conversion matrix.

The process of the proof is similar to the one of the proof of the theorem in the reference (Maeda et al.(1992)) and (Nevel'son et al.(1973b)). However, we use the simultaneous perturbation as against that the usual extended ARM procedure uses the finite difference approximation. We must take this difference into account.

In order to demonstrate the convergence of u_t to u_*, the gain matrix $H_t^{-1} K_t^T$ used in (5) should be positive definite and bounded. On the other hand, we should guarantee the convergence of u_t to u_* in order to prove the convergence of K_t to $f'(u_*)$. These two convergences depend on each other. The detail of the proof is omitted to save the space.

3. NUMERICAL SIMULATION

In this section, we consider a simple multiple-input multiple-output static system to compare an efficiency of the algorithm proposed here and the usual extended ARM algorithm. As described before, this algorithm has an advantage especially in the case that the dimension of the system is higher. In this point of view, we handle the following simple 10 inputs 11 output linear system.

$$z = D u + \psi, \ D = \begin{pmatrix} 1 & 1 & & 0 \\ 1 & 0 & & \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \tag{14}$$

Where, $u \in \Re^{10}$, $z, \psi \in \Re^{11}$. Thus, D is 11×10 matrix. That is,

$$z = \begin{pmatrix} u^{(1)} + u^{(2)} \\ u^{(1)} \\ u^{(2)} \\ \vdots \\ u^{(10)} \end{pmatrix} + \psi \quad (15)$$

We set up a desired value of the output as follows:

$$r = (0\ 1\ 2\ 0\ \cdots\ 0)^T \quad (16)$$

Then, we obtain the evaluation function from Eq.(4).

$$\begin{aligned} J(u) &= (u^{(1)} + u^{(2)})^2 + (u^{(1)} - 1)^2 \\ &\quad + (u^{(2)} - 2)^2 \\ &\quad + (u^{(3)})^2 + \cdots + (u^{(10)})^2 \\ &\quad + \mathrm{E}(\psi^T \psi) \end{aligned} \quad (17)$$

From $\partial J / \partial u = 0$, we know the following least-square input parameters:

$$u_*^{(1)} = 0,\ u_*^{(2)} = 1,\ u_*^{(3)} = 0, \cdots,\ u_*^{(10)} = 0$$

Initial value $u_1 = (5.0\ \cdots\ 5.0)^T$. Moreover,

$$K_1 = \begin{pmatrix} 5.0 & & 0 & \\ & \ddots & & \\ & & 5.0 & \\ 0 & & & 0.0 \end{pmatrix} \quad (18)$$

The observation noise ψ is a uniform random number $[-0.1\ 0.1]$ with zero mean. The perturbation is also a uniform random number $[-1.0\ 1.0]$ with zero mean, but we remove $[-0.1\ 0.1]$ to ensure the condition [7]. Sequence c_t is $t^{-0.3333}$.

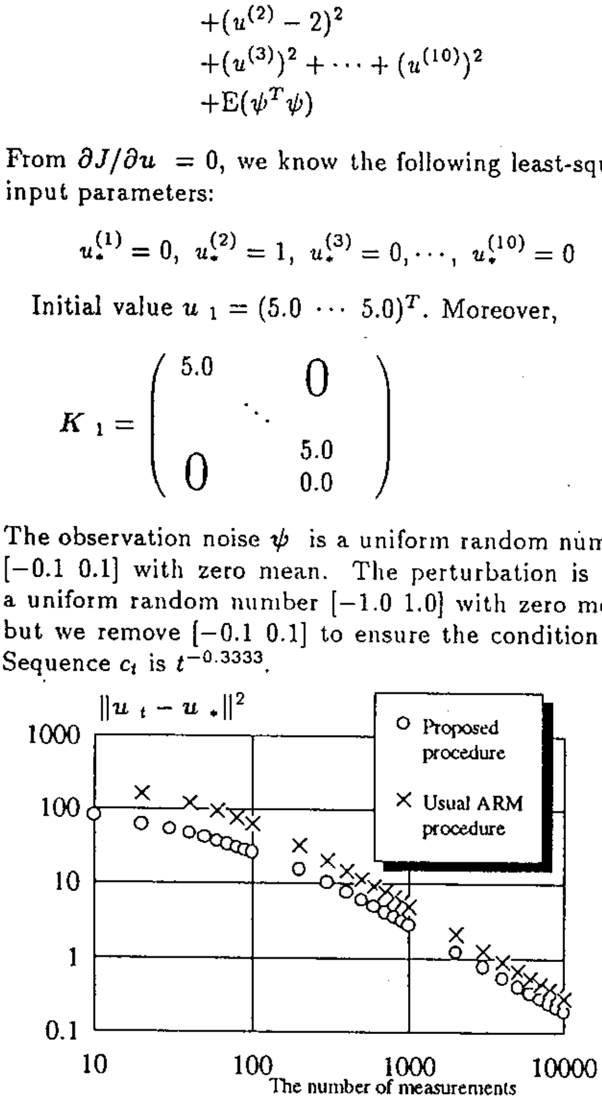

Fig. 1. Simulation results.

Figure 1 shows a simulation result by the proposed algorithm (11) and (12) and by the usual extended ARM procedure. On average of ten trials, this figure shows an outcome by numerical simulations. In this figure, the horizontal axis denotes the number of measurements of the system. The vertical axis denotes the squared error $\|u_t - u_*\|^2$.

The convergence rate of the algorithm proposed here is faster than the one of the usual extended ARM procedure. As we described before, the advantage of this algorithm is outstanding as a system is larger.

4. CONCLUSION

In this paper, we proposed an extended ARM algorithm via the simultaneous perturbation. Our algorithm is applicable to the case that the dimension of the output is greater than the one of the input. This algorithm recursively estimates the least-square approximation input parameter. Moreover, even if the dimension of the system increases, there is no need to increase the number of measurements.

The author provided applications of ARM-like procedure to few area (Maeda et al.(1990a), Maeda et al.(1990b), Maeda et al.(1990c)). The algorithm proposed here is also applicable to these fields.

REFERENCES

Maeda,Y. and Kawamura,Y.(1990a) On-Line System Optimization Using Extended Adaptive Robbins-Monro Procedure, The Trans.(A) of the Institute of Electronics, Information and Communication Engineers of Japan, J73-A, 3, 626-631(in Japanese).

Maeda,Y. and Kawamura,Y.(1990b) Optimization Using the Adaptive Robbins-Monro Procedure for Linear Unknown Stochastic Systems, The Trans.(A) of the Institute of Electronics, Information and Communication Engineers of Japan, J73-A, 4, 783-789(in Japanese)

Maeda,Y. and Kawamura,Y.(1990c) Learning Control via the Adaptive Robbins-Monro Procedure, Trans. of the Institute of Systems, Control and Information Engineers, 3, 7, 226-228(in Japanese)

Maeda,Y. and Kanata,Y.(1991) An Algorithm for a Least-Square Approximation Problem of Unknown Systems, Proceedings of 1991 International Conference on Industrial Electronics, Control and Instrumentation, 3, 1881-1886

Maeda,Y. and Kawamura,Y.(1992) An Extension of the Adaptive Robbins-Monro Procedure for a Least-Square Approximation Problem of Unknown Systems, Electronics and Communication in Japan,Part3, SCRIPTA TECHNICA, INC., 75, 4, 101-110

Nevel'son,M. B. and Has'minskii,R. Z.(1973a) Stochastic Approximation and Recursive Estimation, American Mathematical Society

Nevel'son,M.B. and Has'minskii,R.Z.(1973b) An Adaptive Robbins-Monro Procedure, Automation and Remote control, 34, 10, 1594-1607

Spall,J. C.(1992) Multivariate Stochastic Approximation Using a Simultaneous Perturbation Gradient Approximation, IEEE Trans. on Automatic control, 37, 3, 332-341