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Discrete Optimization via SPSA

László Gerencsér
Computer and Automation Institute
Hungarian Academy of Sciences
Kende 13-17, Budapest, 1111, HUNGARY

Stacy D. Hill
Applied Physics Laboratory
John Hopkins University
Laurel, MD 20723-6099, U.S.A.

Zsuzsanna Vágó
Computer and Automation Institute
Hungarian Academy of Sciences
Kende 13-17, Budapest, 1111, HUNGARY

Abstract

We consider the use of a fixed gain version of SPSA (simultaneous perturbation stochastic approximation) for optimizing a class of discrete functions. The procedure has been modified to obtain an optimization method that is applicable to cost functions defined on a grid of points in Euclidean p-space having integer components. We discuss some related results on fixed gain SPSA and describe an application of the method to a resource allocation problem.

1. INTRODUCTION

The *simultaneous perturbation stochastic approximation* (SPSA) method [9] is a tool for solving optimization problems in which the cost function is analytically unavailable or difficult to compute. The method is essentially a randomized version of the Kiefer-Wolfowitz method in which the gradient is estimated using only two measurements of the cost function at each iteration. SPSA is particularly efficient in problems of high dimension and where the cost function must be estimated through expensive simulations. The convergence properties of the algorithm have been established in a series of papers ([2], [5], [6], [9]).

The present paper discusses a modification [7] of SPSA for discrete optimization. In particular we consider an optimization problem where the cost function is an expectation and is defined only on the grid Z^p of points in R^p having integer components. The main motivation for the algorithm is a class of resource allocation problems, which arise in a variety of applications that include, for example, the problems of distributing search effort to detect a target, allocating buffers in a queueing network, and scheduling data transmission in a communication network.

2. PROBLEM FORMULATION

Consider the problem of finding the minimum of a real-valued function $L(\theta)$, for $\theta \in D$, where D is an open domain in R^p. The function is not assumed to be explicitly known, but noisy measurements $M(n,\theta)$ of it are available:

$$M(n,\theta) = L(\theta) + \varepsilon_n(\theta)$$

where $\{\varepsilon_n\}$ is the measurement noise process. We assume that the function $L(\cdot)$ is at least three-times continuously differentiable and has a unique minimizer in D. The process $\{\varepsilon_n\}$ is a zero-mean process, uniformly bounded and smooth (in θ) in an appropriate technical sense ([4]). The problem is to minimize $L(\cdot)$ using only the noisy measurements $M(\cdot)$.

The *simultaneous perturbation stochastic approximation* (SPSA) algorithm for minimizing functions relies on the *simultaneous perturbations* (SP) gradient approximation. At each iteration k of the algorithm, we take a random *perturbation* vector $\Delta_k = (\Delta_{k1},\ldots,\Delta_{kp})^T$, where the Δ_{ki}'s form an i.i.d. sequence of Bernoulli random variables taking the values ± 1. The perturbations are assumed to be independent of the measurement noise process. In fixed gain SPSA, the step size of the perturbation is fixed at, say, some $c > 0$. To compute the gradient estimate at iteration k, we evaluate $M(\cdot)$ at two values of θ:

$$M_k^+(\theta) = L(\theta + c\Delta_k) + \varepsilon_{2k-1}(\theta + c\Delta_k)$$

$$M_k^-(\theta) = L(\theta - c\Delta_k) + \varepsilon_{2k}(\theta - c\Delta_k).$$

The i-th component of the gradient estimate is

$$H_i(k,\theta) = \frac{(M_k^+(\theta) - M_k^-(\theta))}{2c\Delta_{ki}}.$$

3. THE FIXED GAIN SPSA METHOD

Let $a > 0$ be fixed. Starting with an initial estimate $\hat{\theta}_1$, we recursively compute a sequence of estimates

$$\hat{\theta}_{k+1} = \hat{\theta}_k - aH\left(k+1, \hat{\theta}_k\right) \qquad (1)$$

The assumed boundedness of the noise and assuming the stability of an associated ODE ensures that the sequence of estimates is bounded. (The pathwise behavior of estimator process generated by fixed gain SPSA can be analyzed using the result of [4].)

Proposition: Suppose that $L(\cdot)$ has a unique minimizing value at, say, θ^*. Under appropriate technical conditions $\left|\hat{\theta}_k - \theta^*\right| \le \delta_k$, where (δ_k) is an L-mixing process.

Assume that θ is restricted to the integer grid in R^p. Suppose that $L(\cdot)$ is convex in the sense that at any point of its graph there is a *supporting* hyperplane. There exists a (sufficiently smooth) extension of L to all of R^p, which we denote by $\tilde{L}$. Apply a suitably defined fixed gain SPSA method to the smooth extension, with the additional requirement that we stay on the grid all the time. To achieve a version of (1) that evolves on the grid, take $a = 1$ and replace H with its truncation $\lfloor H \rfloor$, where $\lfloor H \rfloor = (\lfloor H_1 \rfloor, \ldots, \lfloor H_p \rfloor)$, where for $\lfloor y \rfloor$ denotes the integer that is closest to y for each $y \in R$.

Suppose that $L(\cdot)$ has a unique minimizing value θ^*. Assume there is (smooth) strictly convex extension $\tilde{L}$. The extension provides an estimate of the tracking error $\left|\hat{\theta}_k - \theta^*\right|$, where the SA sequence $\{\hat{\theta}_k\}$ is obtained from the extension. Suppose that $\tilde{L}$ has a unique minimizing value $\tilde{\theta}^*$. From the proposition, $\left|\hat{\theta}_k - \tilde{\theta}^*\right|$ is bounded by an L-mixing process; hence $E\left(\left|\hat{\theta}_k - \theta^*\right|\right)$ is uniformly bounded.

3. RESOURCE ALLOCATION

The discrete version of SPSA is motivated by a class of multiple discrete resource allocation problems ([1], [7], [8]). The goal is to distribute a finite amount of resources of different types to finitely many classes of users, where the amount of resources that can be allocated to any user class is discrete. There are n types of *resources*, where N_j denotes the number of resources of type j, $j = 1, \ldots, n$. These resources are to be allocated over M *user* classes. Let θ_{jk} denote the number of resources of type j that are allocated to user class k, and θ be the vector consisting of all the θ_{jk}'s. The allocation of resources to users in class k is denoted θ_k, thus $\theta_k = (\theta_{1k}, \ldots, \theta_{nk})$. For each allocation vector θ there is an associated cost function or performance index $L(\theta)$, which is the expected cost. The goal is to distribute the resources in such a way that cost is minimized:

$$\text{minimize } L(\theta)$$

$$\text{subject to} \sum_{k=1}^{M} \theta_{jk} = N_j, \quad \theta_{jk} \ge 0, \quad 1 \le j \le n$$

The case of interest, of course, is when the cost is observed with noise and the expected cost $L(\theta)$ is non-separable and convex.

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