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RANDOMIZED-DIRECTION STOCHASTIC APPROXIMATION ALGORITHMS USING DETERMINISTIC SEQUENCES

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ABSTRACT

We study the convergence and asymptotic normality of a generalized form of stochastic approximation algorithm with deterministic perturbation sequences. Both one-simulation and two-simulation methods are considered. Assuming a special structure of deterministic sequence, we establish sufficient condition on the noise sequence for a.s. convergence of the algorithm. Construction of such a special structure of deterministic sequence follows the discussion of asymptotic normality. Finally we discuss ideas on further research in analysis and design of the deterministic perturbation sequences.

1 INTRODUCTION

The multivariate version of Kiefer and Wolfowitz's algorithm introduced in Blum (1954) has been a popular approach to solving a high dimensional optimization problem where no estimator of the gradient of the criterion function is available. However this algorithm requires $2p$ simulations at each iteration for a $p-$ dimension objective function. This requirement can incur prohibitively high computational costs in the case where the dimension of the problem is high and expensive simulations are necessary to obtain each measurement. To circumvent the problem, two classes of randomized-direction stochastic approximation algorithms have been proposed: the *randomized direction Kiefer-Wolfowitz* (RDKW) algorithms (Kushner and Clark 1978, Styblinski and Tang 1990), and the *simultaneous perturbation stochastic approximation* (SPSA) algorithms presented in Spall (1992). Typically SPSA or RDKW algorithms *randomly* perturbs all parameter components in *two* parallel simulations at each iteration for any $p-$ dimensional problem. An SPSA requiring only *one* simulation at each iteration has also been proposed in Spall (1997). These algorithms all rely on proper randomization to avoid the large number simulations required in each iteration, and at the same time move along the gradient descent direction on the average. Similar in spirit to the use of low-discrepancy sequences in quasi Monte Carlo integration (Niederreiter 1992), applications of *deterministic* sequences in randomized direction SA have been investigated recently with some success, including Sandilya and Kulkarni (1997) for a two-simulation RDKW algorithms and Bhatnagar et al. (2002) for two-timescale SPSA algorithms. The numerical simulations results reported in Bhatnagar et al. (2002) are particularly encouraging in that significant performance advantages over the random Bernoulli perturbation sequences were consistently observed. In this paper, we present a generalized form of the stochastic approximation algorithm of which SPSA and RDKW are just special cases. Both one-simulation (1D) and two-simulation (2D) forms are considered. In Section 2, with the deterministic sequence assuming a specified structure, we give sufficient conditions for a.s. convergence of both 1D and 2D. In the same section asymptotic normality of both algorithms are also discussed where the structure of deterministic sequence is a little more specified. In Section 3, we discuss how to construct such a specified deterministic perturbation sequence and the principle of defining parameters for practical simulation. Finally, Section 4 offers some concluding remarks. All the proofs will be provided in the Appendix.

2 A.S. CONVERGENCE AND ASYMPTOTIC NORMALITY

Throughout the paper, we will consider the problem of locating minimum of a function $L : \mathbb{R}^p \to \mathbb{R}$. We assume that L satisfies the following conditions.

(A1) The gradient of L, denoted by $g = \nabla L$, exists and is uniformly continuous.

(A2) There exist $\theta^* \in \mathbb{R}^p$ such that

- $f(\theta^*) = 0$; and
- for all $\delta > 0$, there exists $h_\delta > 0$ such that $\|\theta - \theta^*\| \geq \delta$ implies $f(\theta)^T(\theta - \theta^*) \geq h_\delta \|\theta - \theta^*\|^2$.

We rely mainly on the following convergence theorem from Wang et al. (1996), Wang et al. (1997) and lemma 2.2 to derive sufficient conditions on the perturbations and noise.

Theorem 2.1: Consider the stochastic approximation algorithm

$$\theta_{n+1} = \theta_n - a_n g(\theta_n) + a_n e_n + a_n b_n, \tag{1}$$

where $\{\theta_n\}$, $\{e_n\}$, and $\{b_n\}$ are sequences on $\mathbb{R}^p$, $g \colon \mathbb{R}^p \to \mathbb{R}^p$ satisfies Assumption (A2), $\{a_n\}$ is a sequence of positive real numbers satisfying $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$, and $\lim_{n\to\infty} b_n = 0$. Suppose that the sequence $\{g(\theta_n)\}$ is bounded. Then, for any θ_1 in $\mathbb{R}^p$, $\{\theta_n\}$ converges to θ^* if and only if $\{e_n\}$ satisfies any of the following conditions:

(B1)

$$\lim_{n\to\infty} \left(\sup_{n \leq k \leq m(n,T)} \left\| \sum_{i=n}^{k} a_i e_i \right\| \right) = 0$$

for some $T > 0$, where $m(n, T) \triangleq \max\{k : a_n + \cdots + a_k \leq T\}$.

(B2)

$$\lim_{T\to 0} \frac{1}{T} \limsup_{n\to\infty} \left(\sup_{n \leq k \leq m(n,T)} \left\| \sum_{i=n}^{k} a_i e_i \right\| \right) = 0.$$

(B3) For any $\alpha, \beta > 0$, and any infinite sequence of non-overlapping intervals $\{I_k\}$ on $\mathbb{N}$ there exists $K \in \mathbb{N}$ such that for all $k \geq K$,

$$\left\| \sum_{n \in I_k} a_n e_n \right\| < \alpha \sum_{n \in I_k} a_n + \beta.$$

(B4) There exist sequences $\{f_n\}$ and $\{g_n\}$ with $e_n = f_n + g_n$ for all n such that

$$\sum_{k=1}^{n} a_k f_k \text{ converges, and } \lim_{n\to\infty} g_n = 0.$$

(B5) The weighted average $\{\bar{e}_n\}$ of the sequence $\{e_n\}$ defined by

$$\bar{e}_n = \frac{1}{\beta_n} \sum_{k=1}^{n} \gamma_k e_k,$$

converges to 0, where

$$\begin{aligned} \beta_n &= \begin{cases} 1 & n = 1, \\ \prod_{k=2}^{n} \frac{1}{1-a_k} & \text{otherwise,} \end{cases} \\ \gamma_n &= a_n \beta_n. \end{aligned}$$

Proof. See Wang et al. (1996) for a proof for conditions (B1–4) and Wang et al. (1997) for a proof for condition (B5). □

Lemma 2.2: Let $\{a_n\}$, $\{b_n\}$ and $\{e_n\}$ be sequences in $\mathbb{R}$ and $\{r_n\}$ in $\mathbb{R}^p$ such that:

(C1) $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} \frac{a_n}{c_n} = 0$, $\sum_{n=1}^{\infty} a_n = \infty$;

(C2) $S_0 = \sup_{n,m} \left\| \sum_{i=n}^{m} r_i \right\| < \infty$, $E_0 = \sup_n \|e_n\| < \infty$;

(C3) $\sum_{n=1}^{\infty} |\frac{a_n}{c_n} - \frac{a_{n+1}}{c_{n+1}}| < \infty$ or $\lim_{n\to\infty} \frac{1}{c_n} - \frac{a_{n+1}}{a_n c_{n+1}} = 0$;

(C4) $\{\frac{\|e_n - e_{n+1}\|}{c_n}\}$ satisfies condition (B1-5).

Then $\{\frac{r_n e_n}{c_n}\}$ satisfies condition (B1).

Proof. See appendix. □

Lemma 2.2 still holds if $\{r_n\}$ and $\{e_n\}$ are in $\mathbb{R}^{p\times p}$ and $\mathbb{R}^p$, respectively. It is trivial to show that the first alternative of (C3) can be achieved by assuming $\frac{a_n}{c_n} \downarrow 0$.

We describe a generalized form of the stochastic approximation algorithm. Let $\{d_n\}$ and $\{r_n\}$ are deterministic sequences on $\mathbb{R}^p$ and we denote the ith component of d_n and r_n as d_{ni} and r_{ni}, respectively . The recursive formulae of one-simulation and two-simulation forms are:

(1D)

$$\theta_{n+1} = \theta_n - a_n \frac{y_n^+}{c_n} r_n, \tag{2}$$

(2D)

$$\theta_{n+1} = \theta_n - a_n \frac{y_n^+ - y_n^-}{2c_n} r_n \tag{3}$$

where y_n^+ and y_n^- are noisy samples obtained from simulations of the function L at perturbed points, defined by

$$y_n^+ = L(\theta_n + c_n d_n) + e_n^+, \quad y_n^- = L(\theta_n - c_n d_n) + e_n^-,$$

with additive noise e_n^+ and e_n^-, respectively.

Obviously if $\{d_n\}$ and $\{r_n\}$ coincide, the two-simulation algorithm defined by (3) would reduce to the RDKW algorithms. SPSA is defined when $\{d_n\}$ and $\{r_n\}$ are related by $d_n = [\frac{1}{r_{n1}}, \cdots, \frac{1}{r_{np}}]^T$.

Our main results can be presented as four propositions. The first two discuss a.s. convergence of $\{\theta_n\}$ defined by (2) and (3), respectively. The next two propositions give asymptotic normality of $\{\theta_n\}$ of both cases. Note we always assume $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} c_n = 0$, $\lim_{n\to\infty} \frac{a_n}{c_n} = 0$ and $\sum_n a_n = \infty$.

Proposition 2.3 (convergence of one-simulation algorithm): Suppose that the Assumptions (A1–2) hold, and

(D1) $\sum_{n=1}^{\infty} |a_n - a_{n+1}| < \infty$ or $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$

(D2) $\sum_{n=1}^{\infty} |\frac{a_n}{c_n} - \frac{a_{n+1}}{c_{n+1}}| < \infty$ or $\lim_{n\to\infty} \frac{1}{c_n} - \frac{a_{n+1}}{a_n c_{n+1}} = 0$

(D3) $L(\theta_n)$ and $g(\theta_n)$ are bounded;

(D4) both $\{d_n\}$ and $\{r_n\}$ are periodical with period M, $\sum_{n=1}^{M} r_n = 0$ and $\frac{1}{M}\sum_{n=1}^{M} r_n d_n^T = \rho I$, where $\rho > 0$;

(D5) $\{\frac{a_n}{c_n^2}\}$ satisfies condition (B1-5), both $\{\frac{e_n^+ r_n}{c_n}\}$ and $\{\frac{a_n|e_n|}{c_n^2}\}$ satisfy condition (B1-5) a.s.

Then, $\{\theta_n\}$ defined by (2) converges to θ^* a.s.

Proof. See appendix. □

The boundedness of L and g are not very strong. Practically we often restrict $\{\theta_n\}$ to a compact set by doing projection. Then uniform continuity and boundedness are implied by continuity. The assumption $\sum_{n=1}^{M} r_n d_n^T = \rho I$ implies that $p = Rank(\sum_{n=1}^{M} r_n d_n^T) \le \sum_{n=1}^{M} Rank(r_n d_n^T) = M$. Actually we can see from proof that $\{r_n\}$ and $\{d_n\}$ does not have to be periodical, all we need is that

- the partial sum of $\{r_n\}$ is bounded;
- there exists a positive constant ρ such that the partial sum of $\{r_n d_n^T - \rho I\}$ is bounded.

Proposition 2.4 (convergence of two-simulation algorithm): Suppose that the Assumptions (A1–2,D1) hold, and

- $\{g(\theta_n)\}$ is bounded, $\lim_{n\to\infty} c_n = 0$;
- $\{d_n\}$ is periodical with period M, and $\frac{1}{M}\sum_{n=1}^{M} r_n d_n^T = \rho I$, where $\rho > 0$.

Then, $\{\theta_n\}$ defined by (3) converges to θ^* a.s if and only if $\frac{e_n^+ r_n}{c_n}$ satisfies (B1-5)a.s.

Proof. Proof completes by following the same arguments in the proof of Proposition 2.3. □

We denote the Hessian matrix and sth derivative of $L(\theta)$ as $H(\theta)$ and $L^{(3)}(\theta)$ respectively.

Proposition 2.5 (asymptotic normality of one-simulation algorithm): Suppose that the Assumptions (A1–2) hold and $\{\theta_n\}$ is defined by (2), and

(E1) $a_n = a/n^\alpha$ and $c_n = c/n^\gamma$ where $a, c, \alpha, \gamma > 0$;

(E2) $\alpha \le 1$, $\beta = \alpha - 2\gamma > 0$, $3\gamma - \alpha/2 \ge 0$, $1 + 2\gamma < 2\alpha$;

(E3) both $\{d_n\}$ and $\{r_n\}$ are periodical with period M, $\sum_{n=1}^{M} r_n = 0$; $\sum_{n=1}^{M} r_n \otimes d_n \otimes d_n = 0$; $\frac{1}{M}\sum_{n=1}^{M} r_n d_n^T = \rho I$, where $\rho > 0$,

(E4) $Q \equiv M^{-1}\sum_{n=1}^{M} r_n r_n^T$ and orthogonal matrix P satisfies $P^T H(\theta^*) P = (a\rho)^{-1}$ diag$(\lambda_1, \cdots, \lambda_p)$;

(E5) L, g, H and $L^{(3)}$ are all continuous and bounded;

(E6) $\lim_{n\to\infty} n^{-\beta} e_n^+ = 0$, $E(e_n^+|\mathcal{F}_n) = 0$ a.s. and $E((e_n^+)^2|\mathcal{F}_n) \to \sigma^2$ a.s., $\forall n$, where $\mathcal{F}_n \equiv \{\theta_0, \theta_1, \cdots, \theta_n\}$;

(E7) there exists $\delta > 0$ such that $\sup_n E|e_n^+|^{2+2\delta} < \infty$.

Then $n^{\beta/2}(\theta_n - \theta^*) \overset{dist}{\to} N(\mu, PXP^T)$, as $n \to \infty$, where $X_{ij} = a^2 c^{-2} \sigma^2 [P^T Q P]_{ij} (\lambda_i + \lambda_j - \beta_+)^{-1}$ with $\beta_+ = \beta < 2min_i\lambda_i$ if $\alpha = 1$ and $\beta_+ = 0$ if $\alpha < 1$,and

$$\mu = \begin{cases} 0 & \text{if } 3\gamma - \alpha/2 > 0, \\ (a\rho H(\theta^*) - \frac{1}{2}\beta_+ I)^{-1} T & \text{if } 3\gamma - \alpha/2 = 0, \end{cases}$$

where the lth element of T is

$$-\frac{ac^2}{6M}[L_{lll}^{(3)}(\theta^*)\sum_{n=1}^{M} d_{nl}^3 r_{nl} + 3\sum_{i=1, i\ne l}^{M} L_{lii}^{(3)}(\theta^*)\sum_{n=1}^{M} d_{ni}^2 d_{nl} r_{nl} + 6\sum_{i,j=1; i\ne j\ne l\ne i}^{M} L_{lij}^{(3)}(\theta^*)\sum_{n=1}^{M} d_{nl} d_{ni} d_{nj} r_{nl}].$$

Proof. See appendix. □

Proposition 2.6 (asymptotic Normality of two-simulation algorithms): Suppose that the Assumptions (A1–2, E1–3) hold and $\{\theta_n\}$ is defined by (3), and

- both $\{d_n\}$ and $\{r_n\}$ are periodical with period M, let $\frac{1}{M}\sum_{n=1}^{M} r_n d_n^T = \rho I$, where $\rho > 0$, and let orthogonal matrix P such that $P^T H(\theta^*) P = (a\rho)^{-1}$ diag$(\lambda_1, \cdots, \lambda_p)$;
- g and H bounded, $L^{(3)}$ is continuous at θ^*;
- $E(e_n^+ - e_n^- | \mathcal{F}_n) = 0$ a.s. and $E((e_n^+ - e_n^-)^2 | \mathcal{F}_n) \to 4\sigma^2$ a.s., $\forall n$, where $\mathcal{F}_n \equiv \{\theta_0, \theta_1, \cdots, \theta_n\}$;
- there exists $\delta > 0$ such that $\sup_n E|e_n^{(\pm)}|^{2+2\delta} < \infty$.

Then we have the same conclusion as Proposition 2.5.

Proof. Proof completes by following the same arguments in the proof of Proposition 2.5. □

Remark: If we let each component of r_n and d_n assume ± 1, then we get exactly the same result as Proposition 2 in Spall (1992).

The four propositions above show that deterministic perturbation can do at least as well as randomized perturbation asymptotically.

3 CONSTRUCTION OF DETERMINISTIC SEQUENCES

In this section, we present a general mechanism for construction of deterministic sequences $\{r_n\}$ and $\{d_n\}$ that satisfies conditions required for convergence of algorithms. Since stronger conditions required for convergence of one-simulation algorithms, we focus on constructions of sequences that satisfy the conditions stated in Proposition 2.3. The constructed sequences can be applied to two-simulation algorithms as well. We focus on sequences for RDKW and SPSA algorithms and consider the case where components of r_n and d_n take value from $\{\pm 1\}$. Note that in this case, the two classes of algorithms are identical. It is also clear that we only need to construct either $\{r_n\}$ or $\{d_n\}$ since they are identical as well.

Our constructions are based on the notion of orthogonal arrays (Hedayat et al. 1999). We claim that a desirable deterministic sequence in dimension p can be constructed from any binary (two-level) $N \times k$ orthogonal array with $k \geq p$. We first give the definition of orthogonal arrays: **Definition** (Hedayat et al. 1999) An $N \times k$ array A with entries from $S = \{0, 1, \ldots, s\}$ is said to be an orthogonal array with s levels, strength t and index λ if every $N \times t$ subarray of A contains each t-tuple based on S exactly λ times as a row. We use the notation $OA(N, k, s, t)$ to denote such an array.

For example, an $OA(8, 4, 2, 3)$ is given below

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \tag{4}$$

To construct a desired sequence $\{r_n\}$ in $\mathbb{R}^p$ from an $OA(N, k, 2, t)$ with $k \geq p$, we take the following simple steps:

1. Take any p columns from the orthogonal array to form a $N \times p$ array H.
2. Change all the zero entries in H into -1.
3. Use all the row vectors of H as one period for $\{r_n\}$.

For example, a desired sequence $\{r_n\}$ in $\mathbb{R}^4$ can be constructed from the orthogonal array (4) as

$$\begin{aligned} r_1 &= [-1, -1, -1, -1]^T, \\ r_2 &= [-1, -1, 1, 1]^T, \\ r_3 &= [-1, 1, -1, 1]^T, \\ r_4 &= [-1, 1, 1, -1]^T, \\ r_5 &= [1, -1, -1, 1]^T, \\ r_6 &= [1, -1, 1, -1]^T, \\ r_7 &= [1, 1, -1, -1]^T, \\ r_8 &= [1, 1, 1, 1]^T. \end{aligned}$$

Orthogonal arrays have been applied in many areas including experiment designs, coding theory, and cryptography. A large body of literature exists on construction of orthogonal arrays. Hence the proposed construction provides a large set of deterministic sequences for use in stochastic approximation algorithms for optimization. A particular construction based on Hadamard matrices (Seberry and Yamada 1992) is presented in Bhatnagar et al. (2002).

4 CONCLUSION

In this paper, we present a generalized form of the stochastic approximation algorithm of which SPSA and RDKW are special cases. We establish sufficient conditions on deterministic sequences for convergence of these algorithms. Asymptotic Normality are present to show that deterministic sequences can at least achieve the same asymptotic performance with the random sequences. It remains to be shown that appropriately designed deterministic sequences can yield faster convergence than the random sequences even though encouraging simulation results are available.

APPENDIX

Proof of Lemma 2.2

Proof. Let $S_i \equiv \sum_{j=n}^{i} r_j, \forall i > n-1$ and $S_{n-1} = 0$. Then for all $n \le k \le m(n,T)$,

$$\begin{aligned}
\left\|\sum_{i=n}^{k}\frac{a_i}{c_i}r_ie_i\right\| &= \left\|\sum_{i=n}^{k}\frac{a_i}{c_i}(S_i - S_{i-1})e_i\right\| \\
&= \left\|\frac{a_k}{c_k}S_ke_k + \sum_{i=n}^{k-1}S_i(\frac{a_i}{c_i}e_i - \frac{a_{i+1}}{c_{i+1}}e_{i+1})\right\| \\
&\le \left\|\frac{a_k}{c_k}S_ke_k\right\| + \sum_{i=n}^{k-1}\left\|S_i\frac{a_i}{c_i}(e_i - e_{i+1})\right\| \\
&\quad + \sum_{i=n}^{k-1}\left\|(\frac{a_i}{c_i} - \frac{a_{i+1}}{c_{i+1}})S_ie_{i+1}\right\| \\
&\le S_0E_0|\frac{a_k}{c_k}| + S_0\sum_{i=n}^{k-1}\frac{a_i\,\|(e_i - e_{i+1})\|}{c_i} \\
&\quad + S_0E_0\sum_{i=n}^{k-1}|\frac{a_i}{c_i} - \frac{a_{i+1}}{c_{i+1}}|. \qquad (5)
\end{aligned}$$

1. The first term converges to 0 by assumption (C1).
2. Since $\{\frac{\|e_n - e_{n+1}\|}{c_n}\}$ satisfies condition (B4), we have $\{f_n\}$ and $\{g_n\}$ such that $\frac{\|e_n-e_{n+1}\|}{c_n} = f_n + g_n$, $\sum_{n=1}^{\infty} a_nf_n < \infty$ and $\lim_{n\to\infty} g_n = 0$, then we have

$$\begin{aligned}
\sum_{i=n}^{k-1}\frac{a_i\,\|(e_i - e_{i+1})\|}{c_i} &= \sum_{i=n}^{k-1}a_if_i + \sum_{i=n}^{k-1}a_ig_i \\
&\le \sum_{i=n}^{k-1}a_if_i + T\sup_{i\ge n}\|g_i\| \to 0
\end{aligned}$$

3. $\sum_{i=n}^{k-1}|\frac{a_i}{c_i} - \frac{a_{i+1}}{c_{i+1}}| \to 0$ assuming $\sum_{n=1}^{\infty}|\frac{a_n}{c_n} - \frac{a_{n+1}}{c_{n+1}}| < \infty$; assuming $\lim_{n\to\infty}\frac{1}{c_n} - \frac{a_{n+1}}{a_nc_{n+1}} = 0$, we have

$$\begin{aligned}
\sum_{i=n}^{k-1}|\frac{a_i}{c_i} - \frac{a_{i+1}}{c_{i+1}}| &\le \sup_{i\ge n}|\frac{1}{c_i} - \frac{a_{i+1}}{a_ic_{i+1}}|\sum_{n}a_i \\
&\le \sup_{i\ge n}|\frac{1}{c_i} - \frac{a_{i+1}}{a_ic_{i+1}}| \to 0.
\end{aligned}$$

We are done since each term on RHS of (5) converges to 0 when $n \to \infty$. □

Proof of Proposition 2.3:

Proof. By the mean value theorem, we can rewrite (2)

$$\begin{aligned}
\theta_{n+1} &= \theta_n - \rho a_ng(\theta_n) - a_nr_nd_n^T[g(\theta_n + \lambda_nc_nd_n) \\
&\quad - g(\theta_n)] - a_n[r_nd_n^T - \rho I]g(\theta_n) \\
&\quad - \frac{a_n}{c_n}L(\theta_n)r_n - a_n\frac{e_n^+}{c_n}r_n, \qquad (6)
\end{aligned}$$

where $0 \le \lambda_n \le 1$.

1. Since $\lim_{n\to\infty} g(\theta_n + \lambda_nc_nd_n) - g(\theta_n) = 0$ by the uniform continuity of g and $\lim_{n\to\infty} c_n = 0$, $\{r_nd_n^T[g(\theta_n + \lambda_nc_nd_n) - g(\theta_n)]\}$ satisfies condition (B4). Also, we know $\{g(\theta_n + \lambda_nc_nd_n)\}$ is bounded.
2. Combining boundedness of both $\{g(\theta_n + \lambda_nc_nd_n)\}$ and $\{L(\theta_n)\}$ with assumption (D5), we can check (6) and show

$$\lim_{n\to\infty}\theta_n - \theta_{n+1} = 0 \text{ a.s.}$$

   Thus $\lim_{n\to\infty} g(\theta_n) - g(\theta_{n+1}) = 0$ by uniform continuity of g. $\{[r_nd_n^T - \rho I]g(\theta_n)\}$ satisfies condition (B1) by letting $\{c_n\}$, $\{r_n\}$ and $\{e_n\}$ in Lemma 2.2 be $\{1\}$, $\{r_nd_n^T - \rho I\}$ and $\{g(\theta_n)\}$, respectively.
3. Applying mean value theorem to L, we have

$$\begin{aligned}
|L(\theta_n) - L(\theta_{n+1})| = |g^T[\theta_n + \mu_n\times \\
(\theta_n - \theta_{n+1})](\theta_n - \theta_{n+1})|
\end{aligned}$$

   where $0 \le \mu_n \le 1$. $\lim_{n\to\infty} g[\theta_n + \mu_n(\theta_n - \theta_{n+1})] - g(\theta_n) = 0$ implies boundedness of $\|g[\theta_n + \mu_n(\theta_n - \theta_{n+1})]\|$. Hence,

$$\begin{aligned}
\frac{|L(\theta_n) - L(\theta_{n+1})|}{c_n} &\le M_0\frac{\|\theta_n - \theta_{n+1}\|}{c_n} \\
&\le M_0(M_1\frac{a_n}{c_n^2} + M_2\frac{a_n}{c_n^2}|e_n^+|)
\end{aligned}$$

   where the second inequality is obtained by applying (6) to $\theta_n - \theta_{n+1}$ and using some boundedness condition; $M's$ are positive constants. Since, by assumption (D5), the RHS of above formula satisfies condition (B1), it is trivial to prove the LHS also satisfies condition (B1). Hence we can let $\{e_n\}$ in Lemma 2.2 be $\{L(\theta_n)\}$ and conclude $\{\frac{L(\theta_n)r_n}{c_n}\}$ satisfies condition (B1).
4. $\{\frac{e_n^+}{c_n}r_n\}$ satisfies condition (B1) by assumption (D5).

The proof completes by combining above arguments with Theorem 2.1. □

Proof of Proposition 2.5:

Proof. It is easy to show both $\sum_{i=1}^{n} \frac{a_n}{c_n} e_n^+ r_n$ and $\sum_{i=1}^{n} \frac{a_n^2}{c_n^2}(|e_n^+| - E(|e_n^+|)|\mathcal{F}_n)$ are martingales with finite L^2 norm. Hence $\sum_{n=1}^{\infty} \frac{a_n}{c_n} e_n^+ r_n < \infty$ and $\sum_{i=1}^{n} \frac{a_n^2}{c_n^2}|e_n^+| < \infty$ a.s. by L^2 convergence theorem for martingale. Then Proposition 2.2 guarantees the a.s. convergence of θ_n to θ^*. To show the asymptotic normality, we will check if conditions (2.2.1–3) of Fabian (1968) hold. We will use notation of Fabian (1968) as well. Let $0 \le \lambda_n, \eta_n \le 1$. Use mean value Theorem and rewrite (2):

$$\begin{aligned}\theta_{n+1} &= \theta_n - a_n r_n d_n^T g(\theta_n) - \frac{a_n}{c_n} L(\theta_n) r_n \\ &\quad - a_n \frac{e_n^+}{c_n} r_n - \frac{1}{2} a_n c_n r_n d_n^T H(\theta_n) d_n \\ &\quad - \frac{1}{6} a_n c_n^2 r_n L^{(3)}(\theta_n + \lambda_n c_n d_n) d_n \otimes d_n \otimes d_n.\end{aligned} \tag{7}$$

Use this formula M times, we have

$$\begin{aligned}\theta_{nM+M} - \theta^* &= (I - n^{-\alpha}\Gamma_n)(\theta_{nM} - \theta^*) \\ &\quad + n^{-(\alpha+\beta)/2}\Phi_n V_n \\ &\quad + n^{-\alpha-\beta/2}(T_n^{(1)} + T_n^{(2)} + T_n^{(3)} + T_n^{(4)}),\end{aligned}$$

where

$$\Gamma_n = a \sum_{i=nM}^{nM+M-1} (\frac{i}{n})^{-\alpha} r_i d_i^T H(\theta_{nM} + \eta_n(\theta_{nM} - \theta^*)) \xrightarrow{a.s.} a M^{1-\alpha} \rho H(\theta^*)$$

$$\Phi_n = I$$

$$V_n = \frac{a}{c} \sum_{i=nM}^{nM+M-1} (\frac{i}{n})^{-\alpha+\gamma} e_i^+ r_i$$

$$T_n^{(1)} = -a n^{\beta/2} \sum_{i=nM}^{nM+M-1} (\frac{i}{n})^{-\alpha} r_i d_i^T [g(\theta_i) - g(\theta_{nM})]$$

$$T_n^{(2)} = -\frac{a}{c} n^{\alpha/2} M^{-\alpha+\gamma} \sum_{i=nM}^{nM+M-1} (\frac{i}{nM})^{-\alpha+\gamma} L(\theta_i) r_i$$

$$T_n^{(3)} = -\frac{1}{2} a c n^{\alpha/2-2\gamma} \sum_{i=nM}^{nM+M-1} (\frac{i}{n})^{-\alpha-\gamma} r_i d_i^T H(\theta_i) d_i$$

$$T_n^{(4)} = -\frac{a c^2 n^{\alpha/2-3\gamma}}{6} \times \sum_{i=nM}^{nM+M-1} (\frac{i}{n})^{-\alpha-2\gamma} r_i L^{(3)}(\theta_i + \lambda_i c_i d_i) \cdot d_i \otimes d_i \otimes d_i.$$

To prove $T_n^{(2)} \xrightarrow{L^2} 0$, we have

$$\begin{aligned}T_n^{(2)} &= K_0 n^{\alpha/2} \sum_{i=nM}^{nM+M-1} ((\frac{i}{nM})^{-\alpha+\gamma} - 1) L(\theta_i) r_i \\ &\quad + K_0 n^{\alpha/2} \sum_{i=nM}^{nM+M-1} L(\theta_i) r_i \\ &= O(n^{-\alpha/2+\gamma}) + K_0 n^{\alpha/2} \sum_{i=nM}^{nM+M-1} (L(\theta_i) - L(\theta_{nM})) r_i \\ &= o(1) + K_0 n^{\alpha/2} \sum_{i=nM}^{nM+M-1} (\theta_i - \theta_{nM})^T g(\theta'_{nM}) \\ &= o(1) + n^{\alpha/2} O(n^{-\alpha+\gamma}) \\ &= o(1).\end{aligned}$$

The second equality is by $(1 + \frac{A}{n})^{-\alpha+\gamma} - 1 = O(1/n)$ and $\sum_{i=nM}^{nM+M-1} r_i = 0$; the third is by taking Taylor expansion and θ'_{nM} is on the line segment between θ_i and θ_{nM}; the fourth is by applying (7) to $\theta_i - \theta_{nM}$. Of course boundedness of functions are required when necessary. Also, $o(\cdot)$ and $O(\cdot)$ are in terms of L^2 norm and K_0 is a constant.

We have shown that $T_n^{(2)} \xrightarrow{L^2} 0$. Actually similar argument can be used to show that $T_n^{(1)} \xrightarrow{L^2} 0$ and $T_n^{(3)} \xrightarrow{L^2} 0$. If $3\gamma - \alpha/2 > 0$, we can also show $T_n^{(4)} \xrightarrow{L^2} 0$. If $3\gamma - \alpha/2 = 0$, it is easy to show that $T_n^{(4)} \xrightarrow{a.s.} M^{1-\alpha-\beta/2} T$

It is easy to show that $E_{\mathcal{F}_n} V_n = 0$ and $E_{\mathcal{F}_n} V_n V_n^T \xrightarrow{L^2} \frac{a^2\sigma^2}{c^2} M^{1-2\alpha+2\gamma} Q$. To show

$$\lim_{k\to\infty} E(\chi_{\|V_n\|^2 \ge r n^\alpha} \|V_n\|^2) = 0, \forall r > 0$$

we have

$$\begin{aligned}E(\chi_{\|V_n\|^2 \ge r n^\alpha} \|V_n\|^2) &\le P(\|V_n\|^2 \ge r n^\alpha)^{\delta'/(1+\delta')} \\ &\quad \cdot (E\|V_n\|^{2+2\delta'})^{1/(1+\delta')} \\ &\le K_1 (\frac{E\|V_n\|^2}{r n^\alpha})^{\delta'/(1+\delta')} \\ &\le K_2 n^{-\alpha\delta'/(1+\delta')} \to 0\end{aligned}$$

where K_1 and K_2 are constants and $0 < \delta' < \delta$.

Since all the conditions (2.1.1–3) in Fabian (1968) are verified, we have

$$n^{\beta/2}(\theta_{nM} - \theta^*) \stackrel{dist}{\to} N(M^{-\beta/2}\mu, M^{-\beta}PXP).$$

That is,

$$(nM)^{\beta/2}(\theta_{nM} - \theta^*) \stackrel{dist}{\to} N(\mu, PXP).$$

For all $0 < i < M$, we can similarly prove

$$(nM+i)^{\beta/2}(\theta_{nM+i} - \theta^*) \stackrel{dist}{\to} N(\mu, PXP).$$

□

REFERENCES

Bhatnagar, S., M. C. Fu, S. I. Marcus, and I-J. Wang. 2002. Two-timescale simultaneous perturbation stochastic approximation using deterministic perturbation sequences. Submitted to *ACM Transactions on Modeling and Computer Simulation*.

Blum, J. R. 1954. Multidimensional stochastic approximation methods. *Ann. Math. Stats.* 25: 737–744.

Fabian, V. 1968. On asymptotic normality in stochastic approximation. *The Annals of Mathematical Statistics* 39(4): 1327–1332.

Hedayat, A. S., N. J. A. Sloane, and J. Stufken. 1999. *Orthogonal Arrays: Theory and Applications*, Springer Verlag, New York, NY.

Kushner, H.J. and D. S. Clark. 1978 *Stochastic Approximation Methods for Constrained and Unconstrained Systems*, Springer Verlag, New York.

Niederreiter, H. 1992. *Random Number Generation and Quasi-Monte Carlo Methods*, SIAM, Philadelphia.

Sandilya, S., and S. R. Kulkarni. 1997. Deterministic sufficient conditions for convergence of simultaneous perturbation stochastic approximation algorithms. *The 9th INFORMS Applied Probability Conference*, Boston, MA.

Seberry, J. and M. Yamada. 1992. Hadamard matrices, sequences, and block designs. In *Contemporary Design Theory – A Collection of Surveys* (eds. D.J. Stinson and J. Dintiz), pp. 431–560, John Wiley and Sons.

Spall, J.C. 1992. Multivariate stochastic approximation using a simultaneous perturbation gradient approximation. *IEEE Transactions on Automatic Control*, 37(3):332-341.

Spall, J.C. 1997. A one-measurement form of simultaneous perturbation stochastic approximation. *Automatica*, 33:109–112.

Styblinski, M. A., T-S. Tang. 1990. Experiments in nonconvex optimization. Stochastic approximation with function smoothing and simulated annealing. *Neural Networks*, 3(4):467–483.

Wang, I-J., E. K. P. Chong, and S. R. Kulkarni. 1996. Equivalent necessary and sufficient conditions on noise sequences for stochastic approximation algorithms. *Advances in Applied Probability*, 28:784–801.

Wang, I-J., E. K. P. Chong, and S. R. Kulkarni. 1997. Weighted averaging and stochastic approximation. *Mathematics of Control, Signals, and Systems*, 10(1): 41–60.

Wang, I-J. and E.K.P. Chong. 1998. A deterministic analysis of stochastic approximation with randomized directions. *IEEE Transactions on Automatic Control*, 43(12):1745–1749.

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