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Discrete optimization, SPSA and Markov Chain Monte Carlo methods

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Abstract

The minimization of a convex function defined over the grid $\mathbb{Z}^p$ is considered. A truncated fixed gain SPSA method is proposed and investigated in combination with devices borrowed from the Markov-Chain Monte-Carlo literature. In particular the performance of the proposed method is improved by choosing suitable acceptance probabilities. A new Markovian optimization problem is formulated to get the best rejection probability and gain. A simulation result is presented.

1 Introduction

In this paper we consider the possibility of applying the SPSA method (see [2], [4], [8]) to discrete optimization problems. In particular we consider optimization problems where the cost function is defined only for p-dimensional integer-valued variables, denoted by $\mathbb{R}^p$. Such problems arise in resource allocation problems, where θ denotes an allocation scheme (cf. [1]).

We use a truncated fixed gain SPSA method. The stability of the method is ensured by a resetting mechanism. The performance of the method is enhanced by a device borrowed from the Markov-Chain Monte-Carlo literature.

Consider the following discrete optimization problem: given a smooth, convex cost-function L: $\mathbb{R}^p \to \mathbb{R}$, find its minimum over $\mathbb{Z}^p$ using function-value evaluations only on $\mathbb{Z}^p$. A *benchmark* problem that we will consider in detail is the quadratic problem when

$$L(\theta) = \frac{1}{2}(\theta - x^*)^T A(\theta - x^*) \quad (1)$$

where $x^* \epsilon \mathbb{R}^p$ is such that not all of its coordinates are integers, and A is a symmetric positive definite matrix.

2 SPSA over a discrete set

To approximate the gradient of L we use simultaneous random perturbations (see [8]). Let θ be the current approximation of θ^* and let k be the iteration time. We take a random vector $\Delta = \Delta_k = (\Delta_{k1}, ..., \Delta_{kp})^T$, where Δ_{ki} is a double sequence of i.i.d. random variables. A standard choice is to take a Bernoulli-sequence, taking values $+1$ or -1 with equal probability $1/2$. We take two measurements at $L(\theta + \Delta)$ and $L(\theta - \Delta)$, and define

$$H(\theta) = \frac{1}{2}\Delta^{-1}\left(L(\theta + \Delta) - L(\theta - \Delta)\right) \quad (2)$$

where Δ^{-1} is the vector with components Δ_i^{-1}. Then the fixed gain truncated SPSA method is defined by:

$$\theta_{n+1} = \theta_n - [aH(n, \theta_n)],$$

where [ ] is the integer part coordinatewise.

A simple variant of the above method is obtained if our starting point is a second order SPSA method developed in [9]. Another alternative method is obtained if we use a different truncation procedure, say

$$\theta_{n+1} = \theta_n - \mathrm{sgn}(aH_n(\theta_n)), \quad (3)$$

where sgn is a generalized sign function.

3 Markov-Chain Monte-Carlo methods

An alternative approach to minimizing L is to use Markov-Chain Monte-Carlo (MCMC) methods, such as Metropolis or Metropolis-Hastings, cf. [10]. They are applicable for general problems where $\mathbb{Z}^p$ is replaced by an abstract set. Let $c > 0$ and index the points of $\mathbb{Z}^p$ by integers and set

$$\pi_i = e^{-cL(i)} / \sum_j e^{-cL(j)}. \quad (4)$$

For large c the probability distribution (π) will be concentrated around points of $\mathbb{Z}^p$ where L is small. Let

$$\alpha_i = e^{-cL(i)}$$

be the unnormalized probabilities. In MCMC we construct an ergodic Markov-chain with invariant distribution (π_i) based on α_i-s without actually performing the normalization in (4).

In the Metrolpois-Hastings method we start with an initial Markov-chain with transition probabilities (q_{ij}) and then define p_{ij} for $i \neq j$ so that

$$\alpha_i p_{ij} = \min(\alpha_i q_{ij}, \alpha_j q_{ji})$$

is satisfied. Writing $p_{ij} = q_{ij}\tau_{ij}$, τ_{ij} is called the acceptance probability.

A natural candidate for the q-chain is the Markov-chain defined by the SPSA method, but then q_{ij} is not known explicitely. A possible way out of this difficulty is to chose the acceptance probability in a heuristic manner. Thus we come to the following method: let $0 < \tau < 1$ be a fixed acceptance probability and let $\tau_{ij} = \tau$ if $L(j) > L(i)$ and $\tau_{ij} = 1$ otherwise. Modify the Markov-chain defined by the fixed gain SPSA method using the above acceptance probability. The resulting Markov-chain will be denoted by $\theta_n(\tau)$. To find the best τ define

$$\lambda(\tau) = \mathrm{E}L(\theta_n(\tau))$$

assuming that $\theta_n(\tau)$ has stationary distribution, and then minimize $\lambda(\tau)$ over τ.

A general methodology for solving similar problems has been given in [7]. However, this procedure requires the explicit knowledge of the transition probabilities which is not available. Thus minimization of $\lambda(\tau)$ requires further research.

4 Simulation results

Simulations have been carried out for randomly generated quadratic function in dimensions $p = 20, 50$. We found, that the choice of the stepsize a is critical for the SPSA-method method: the performance depends largely on the right choice of the gain. This influence is less dramatic in the case of the signed second order SPSA-method method. We have developed a simple adaptive scheme for choosing the gain which significantly improves convergence properties. The most efficient and reliable method seems to be the adaptive signed second order SPSA-method.

In the figure below we plot the value of the cost function vs. the iteration time. The purpose of this experiment was to see the effect of the rejection probability τ. The dimension was 20. The figure contains three graphs for three different values of τ for a second order SPSA method. The best result is plotted by dotted line, while the two other weaker results are plotted by solid and dash-dot lines, respectively. We choose the best gain $a = 0.05$ and the values:

$$\tau = 0.95,\ 0.1,\ 0.8.$$

It is seen that the choice of τ is critical for the performance of the second order SPSA method.

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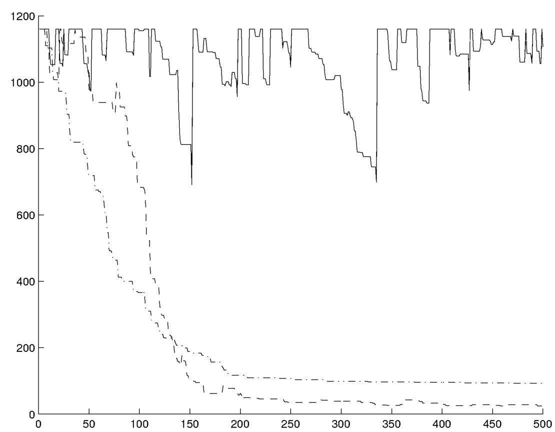

Figure 1: Newton method with different τ-s.